SINGULAR PERTURBATION PROBLEMS IN EARTH-MOON SPACE

by

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THESIS

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ABSTRACT

A program was written for the IBM 360 system of the Naval Postgraduate School Computer Facility to apply the methods of singular perturbation theory to earth-moon trajectories.

Verification of results against previous work was carried out and it was found that agreement could be attained only for energy parameter values of 0.707 or less. No solution for higher values could be found.

The analysis of three-dimensional orbits was then conducted within this restricted range to show the merit of singular perturbation theory as an initial design tool.

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TABLE OF SYMBOLS

Arabic Symbols	
a	Semi major axis of earth elliptic orbit
$\bar{a}$	Semi major axis of moon hyperbolic orbit
b	Semi minor axis of ellipse
D	Earth-moon separation distance
DOCA	Distance of closest approach to moon
e	Eccentricity of outer ellipse
E	Eccentric anomaly
G	Universal Gravitational Constant
h_e	Energy of elliptic orbit about earth
$\bar{h}$	Energy of hyperbolic orbit near moon
i	Inclination from earth-moon plane at earth
$\bar{i}$	Inclination from earth-moon plane at moon
l_e	Angular momentum of elliptic orbit near earth
$\bar{l}$	Angular momentum of hyperbolic orbit near moon
m	Mass
p_e, q_e	Position parameters at earth
p, q	Position parameters at moon
$\bar{r}$	Position vector
T	Phase constant
U	Velocity in x direction
V	Velocity in y direction
x	Independent outer variable

X and $\bar{x}$	Independent inner variables
y and t	Dependent outer variables
$\bar{y}$ and $\bar{t}$	Dependent inner variables

Greek Symbols

γ	Integral of perturbation equation in t
δ	Integral of perturbation equation in y
ξ	Moon's x coordinate
η	Moon's y coordinate
λ	Non-dimensional angular momentum
ρ	Non-dimensional energy
ψ	Angle relative earth after encounter of moon
Ω	Line of nodes in xyz system
$\bar{\Omega}$	Line of nodes in $\bar{x}\bar{y}\bar{z}$ system
$\bar{\theta}$	Apse angle, moon
ω	Apse angle, earth

I. INTRODUCTION

With the advent of space travel and of complex satellite systems, a revival of celestial mechanics has appeared in modern science and engineering. In its present form, commonly referred to as orbital mechanics or astrodynamics, it combines the theories of astronomers and scientists such as Kepler and Newton with modern ideas and technology so that man may navigate his way through space.

Orbital mechanics problems grow in complexity as the number of bodies under consideration grow. In fact, they become so complex that only the problem of two-bodies may be truly solved. Solutions to forms of the three-body problem are accomplished under certain special conditions, but usually numerical approximations exist for these higher order problems. Szebehely [Ref. 1] states,

> "Not only is the two-body problem solved -- and the meaning of 'solution' may be different for astronomers, engineers, and mathematicians -- but a general understanding exists regarding this dynamical system. The problem of three-bodies on the other hand is neither solved nor is the behavior of the dynamical system completely understood.

In order to cope with the problem of space navigation, many methods to determine the motion of a vehicle in space have been attempted. The use of two-body solutions has been used in the method of patched conic sections. Many perturbation techniques have been applied for two, three, four, and more bodies. The most common method of calculating

these trajectories is probably the use of the digital computer. The computer is used to solve the initial-or final-value differential equations of motion in numerical form and is often time consuming and expensive.

Due to the difficulty in finding solutions to trajectory problems, Lagerstrom and Kevorkian in References 2 and 3 formulated a method in which they attempt to treat the restricted three-body problem with a technique similar to that used in certain fields of fluid mechanics. This is the method of solution of singular perturbations by inner and outer expansions and represents an attempt to obtain a uniformly valid analytic expression for trajectory motion in the restricted three-body problem. The work of Lagerstrom and Kevorkian is aimed at giving a tool to more easily predict the effect of initial conditions on the motion of a vehicle as it becomes influenced by a third body in its field of space. This would allow the determination of "ballpark" data for use in more detailed solution programs at a considerable savings of time and expense.

In this paper, an attempt to use the singular perturbation method of Lagerstrom and Kevorkian to study effects of initial conditions on lunar trajectories is made. A brief discussion of Lagerstrom and Kevorkian's theory is followed by some numerical applications of this theory and the results are discussed.

II. THEORY

A. EXAMPLE INNER AND OUTER EXPANSION MATCHING PROBLEM

Before formulating the singular perturbation solution to the restricted three-body problem it is useful to look at a simple example of matched inner and outer asymptotic expansions. The example is basically that of Prandtl and has been used by Van Dyke [4], Ames [5] and others as a classical singular perturbation solution.

If an equation of the form

$$\mu X^{(n)}(t) + F[X, X' \cdots X^{(n-1)}; \mu] = 0$$

is written and a regular expansion is attempted, difficulty is expected since the equation obtained when $\mu = 0$ is of lower order than the original.

Consider the following equation

$$\mu f'' + f' = b \tag{1}$$

with initial conditions,

$$f(0) = 0 \qquad \text{and} \qquad f(1) = 1$$

The exact solution to this equation is

$$f(x:\mu) = (1-b)\left[1 - e^{-x/\mu}\right] / \left[1 - e^{-1/\mu}\right] + bx \tag{2}$$

If a regular perturbation expansion is attempted one may write

$$f = f_0 + \mu f_1 + \mu^2 f_2 + \cdots$$

$$f' = f_0' + \mu f_1' + \mu^2 f_2' + \cdots$$

$$f'' = f_0'' + \mu f_1'' + \mu^2 f_2'' + \cdots$$

substituting into 1

$$\mu[f_0'' + \mu f_1'' + \mu^2 f_2'' + \cdots] + f_0' + \mu f_1' + \mu^2 f_2' + \cdots = b$$

matching powers of μ implies

$$f_0' = b$$

so that by integration

$$f_0 = bx + C \tag{3}$$

The boundary conditions cannot be satisfied unless b = 1.

If the boundary condition at x = 0 is dropped

$$f_0(x,\mu) = (1-b) + bx$$

since

$$f(1) = 1 = b(1) + c \longrightarrow C = 1-b$$

therefore

$$f_0(x,\mu) = bx + (1-b) \tag{4}$$

This approximates the solution quite well except in the region adjacent x = 0 (commonly referred to as the boundary layer), see Figure 1.

Ames states that if terms are lost or boundary conditions discarded in the outer solution they must be included in an inner expansion which must be developed (i.e., this implies matching in the overlap region).

The asymptotic matching principle states that the m-term inner expansion of (the n-term outer expansion) = the n-term outer expansion of (the m-term inner expansion). See Figure 2 for a diagram of the calculation sequence applying this principle.

Returning to the present example, the first term is

$$f_0 = (1-b) + bx$$

for the outer expansion. The first term of the inner expansion is found by a "coordinate stretching." The dependent and the independent variables may be stretched together or separately using physical insight or trial and error, however, the stretching must include essential elements omitted in the outer expansion procedures and must be matchable to the outer expansion.

In the present example one may desire to stretch only the independent variable such that,

$$f(x,\mu) = F(X,\mu)$$

where

$$X = x/s(\mu)$$

Using this in the initial equation

$$\frac{d^2F}{dX^2} + \frac{s(\mu)}{\mu}\frac{dF}{dX} = \frac{s^2(\mu)\,b}{\mu} \tag{5}$$

with

$$F(0) = 0 \quad , \; F(1/s) = 1$$

where

$$s(\mu)\,dX = dx$$

The highest derivative was lost in the outer problem therefore d^2F/dX^2 must be kept for the inner problem. This implies

$$s(\mu)/\mu \neq \infty \qquad \text{as} \qquad \mu \to 0$$

but

$$s(\mu)/\mu = 0 \qquad \text{as} \qquad \mu \to 0$$

implies that the solution satisfying the inner boundary condition is a multiple of X and cannot be matched. Therefore, the only possibility is that $s(\mu)/\mu$ = a constant as $\mu \to 0$, and this constant is assumed to be 1.

This gives

$$\frac{d^2F}{dX^2} + \frac{dF}{dX} = b\mu \tag{6}$$

with

$$F(0) = 0 \; , \; F(1/\mu) = 1 \; , \; s(\mu) = \mu$$

In the inner variables $X = x/\mu$ and F, the following expansions are produced:

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots$$

$$\frac{dF}{dX} = \frac{dF_0}{dX} + \mu \frac{dF_1}{dX} + \mu^2 \frac{dF_2}{dX} + \cdots$$

$$\frac{d^2F}{dX^2} = \frac{d^2F_0}{dX^2} + \mu \frac{d^2F_1}{dX^2} + \mu^2 \frac{d^2F_2}{dX^2} + \cdots$$

which when substituted into 5 yield

$$\frac{d^2F_0}{dX^2} + \frac{dF_0}{dX} = 0 \qquad \text{and} \qquad F(0) = 0$$

and finally,

$$F_0(x) = A(1 - e^{-X}) \tag{7}$$

where the outer boundary condition has been discarded.

To carry out the matching,

1-term outer expansion $= (1-b) + bx$

in inner variables $= (1-b) + b\mu X$

expanded for small μ $= (1-b) + b\mu X$

1 term inner expansion $= (1-b)$

1-term inner expansion $= A(1-e^{-X})$

in outer variables $= A(1-e^{-x/\mu})$

expanded for small μ $= A$

1 term outer expansion $= A$

matching first term outer to first term inner gives

$$A = 1-b$$

therefore

$$F_0 = (1-b)(1-e^{-X})$$

so that now

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots$$

$$F = (1-b)(1-e^{-X}) + \text{Terms}$$

$$f = f_0 + \mu f_1 + \mu^2 f_2 + \cdots$$

$$f = (1-b) + bx + \text{NO MORE TERMS}$$

to find the second term inner expansion

$$\frac{d^2F_1}{dX^2} + \frac{dF_1}{dX} = b$$

$$F_1 = c(e^{-X}+1) + bX$$

therefore

$$F = (1-b)(1-e^{-X}) + \mu\left[c(e^{-X}+1) + bX\right] + \cdots \tag{8}$$

Since the exact solution for the outer variable is known, the matching from 1 to 2 may be skipped as it simply yields,

$$1-b = 1-b$$

Going now from m = 2 and n = 2,

2-term outer expansion	=	$(1-b) + bx$
rewritten in inner variables	=	$(1-b) + \mu bX$
expanded for small μ	=	$(1-b) + \mu bX$
2 term inner expansion	=	$(1-b) + \mu bX$
2-term inner expansion	=	$(1-b)(1-e^{-X}) + \mu\left[c(e^{-X}-1)+bX\right]$
in outer variables	=	$(1-b)(1-e^{-x/\mu}) + \mu\left[c(e^{-x/\mu}-1)+b\frac{x}{\mu}\right]$
expanded for small μ	=	$(1-b) - \mu c + bX + \cdots$
2 term outer expansion	=	$(1-b) + bX - \mu c$
rewrite in inner variables	=	$(1-b) + \mu bx - \mu c$

matching these gives

$$bx = bx - c$$

$$c = 0$$

therefore

$$f = (1-b) + bx$$

$$F = (1-b)(1-e^{-X}) + \mu b X \tag{9}$$

This form of solution is necessary when no single asymptotic expansion is uniformly valid throughout the field of interest in a problem. This problem may arise through factors such as two characteristic lengths as those of the boundary layer thickness near the body compared to the chord length of the body.

This long sample solution is presented above so that as the solution of the earth-moon problem is discussed an understanding of the procedures involved can be felt even though the actual solution is too lengthy to be presented in detail. The procedures are the same in the orbit problem as in the example just discussed but more complicated to carry out.

B. NATURE OF THE PROBLEM

With the method of inner and outer expansions previewed, attention is now turned to the trajectory problem in earth-moon space. It is beneficial to observe first the general problem which must be studied, the three-body problem.

The three-body problem is one aspect of the n-body problem of mechanics. In earth-moon space the problem may

be depicted as in Figure 3. This figure represents the restricted three-body problem which is the second simplest of the many-body problems. The restricted three-body problem, however, is still unsolvable in a closed analytic manner. The only n-body problem which has been solved is the two-body problem. It is therefore of interest to observe the nature of the three-body problem and determine what information one may find from determinable quantities.

Using the system of three masses depicted in Figure 4 it is possible to define three position vectors in space by $\bar{r}_1$, $\bar{r}_2$, and $\bar{r}_3$. Now, if the conventional methods of vector mechanics are applied to the system and the necessary algebraic manipulations are carried out it is possible to determine ten of the necessary eighteen integrals of the total solution to the three-body problem.

Writing the equations of motion with reference to Figure 4

$$
\begin{aligned}
m_1\ddot{\bar{r}}_1 &= G m_1 m_2 \frac{\bar{r}_{12}}{r_{12}^3} + G m_1 m_2 \frac{\bar{r}_{13}}{r_{13}^3} \\
m_2\ddot{\bar{r}}_2 &= G m_1 m_2 \frac{\bar{r}_{21}}{r_{21}^3} + G m_2 m_3 \frac{\bar{r}_{23}}{r_{23}^3} \qquad (10) \\
m_3\ddot{\bar{r}}_3 &= G m_3 m_1 \frac{\bar{r}_{31}}{r_{31}^3} + G m_3 m_2 \frac{\bar{r}_{32}}{r_{32}^3}
\end{aligned}
$$

using the fact that

$$\bar{r}_{ij} = -\bar{r}_{ji} \qquad \text{and} \qquad r_{ij}^2 = r_{ji}^2$$

and adding the three equations

$$m_1\ddot{\bar{r}}_1 + m_2\ddot{\bar{r}}_2 + m_3\ddot{\bar{r}}_3 = G\Big(m_1 m_2 \frac{\bar{r}_{12}}{r_{12}^3} + m_1 m_3 \frac{\bar{r}_{13}}{r_{13}^3} + m_2 m_1\Big(-\frac{\bar{r}_{12}}{r_{12}^3}\Big)$$
$$+ m_2 m_3 \frac{\bar{r}_{23}}{r_{23}^3} + m_3 m_1\Big(-\frac{\bar{r}_{13}}{r_{13}^3}\Big) + m_3 m_2\Big(-\frac{\bar{r}_{23}}{r_{23}^3}\Big)\Big)$$

which yields

$$\sum m_i \ddot{\bar{r}}_i = 0 \qquad (11)$$

which is the conservation of linear momentum.

This vector equation integrated twice then gives six of the ten available integrals.

Now using the relation

$$\bar{r}_i \times \bar{r}_{ij} = \bar{r}_i \times \bar{r}_j - \bar{r}_i \times \bar{r}_i = \bar{r}_i \times \bar{r}_j$$

on equations 10 the following equations are obtained:

$$m_1(\bar{r}_1 \times \ddot{\bar{r}}_1) = G\frac{m_1 m_2}{r_{12}^3}(\bar{r}_1 \times \bar{r}_2) + G\frac{m_1 m_3}{r_{13}^3}(\bar{r}_1 \times \bar{r}_3)$$

$$m_2(\bar{r}_2 \times \ddot{\bar{r}}_2) = G\frac{m_2 m_1}{r_{21}^3}(\bar{r}_2 \times \bar{r}_1) + G\frac{m_2 m_3}{r_{23}^3}(\bar{r}_2 \times \bar{r}_3)$$

$$m_3(\bar{r}_3 \times \ddot{\bar{r}}_3) = G\frac{m_3 m_1}{r_{31}^3}(\bar{r}_3 \times \bar{r}_1) + G\frac{m_3 m_2}{r_{32}^3}(\bar{r}_3 \times \bar{r}_2)$$

If the vector relationship

$$\bar{r}_i \times \bar{r}_j = -\bar{r}_j \times \bar{r}_i$$

is applied and the equations are added the following equation is found:

$$\sum_i m_i(\bar{r}_i \times \ddot{\bar{r}}_i) = G\Big\{\frac{m_1 m_2}{r_{12}^3}\Big[(\bar{r}_1 \times \bar{r}_2) - (\bar{r}_1 \times \bar{r}_2)\Big] + \text{etc}\Big\}$$

which reduces to

$$\sum m_i(\bar{r}_i \times \ddot{\bar{r}}_i) = 0$$

noting that

$$\frac{d}{dt}(\bar{r} \times \dot{\bar{r}}) = \bar{r} \times \ddot{\bar{r}} + \dot{\bar{r}} \times \dot{\bar{r}} = \bar{r} \times \ddot{\bar{r}}$$

the equation

$$\sum m_i(\bar{r}_i \times \ddot{\bar{r}}_i) = 0$$

may be rewritten as

$$\int m_i[\dot{\bar{r}}_i \times \ddot{\bar{r}}_i]dt = m_i[\bar{r}_i \times \dot{\bar{r}}_i]$$

which finally yields

$$\sum m_i[\bar{r}_i \times \bar{v}_i] = \text{Const} \tag{12}$$

which is conservation of angular momentum.

This allows the determination of three more of the necessary integrals for the three-body problem.

The following series of algebraic equations will lead to the determination of the final integral for the three-body problem. Writing Newton's law and operating on it one obtains:

$$\bar{F}_i = m_i \ddot{\bar{r}}_i$$

$$\bar{F}_i \cdot \frac{d\bar{r}_i}{dt} = m_i(\dot{\bar{r}}_i \cdot \ddot{\bar{r}}_i)$$

$$\int \bar{F}_i \cdot dr_i = \frac{m_i \dot{\bar{r}}_i^{\,2}}{2}$$

$$\frac{1}{2} m_i v_i^2 = \int \bar{F}_i \cdot d\bar{r}_i$$

Using this operation on Equation 10

$$\frac{1}{2} m_1 v_1^2 = G m_1 m_2 \int \frac{\bar{r}_{12} \cdot d\bar{r}_1}{r_{12}^3} + G m_1 m_3 \int \frac{\bar{r}_{13} \cdot d\bar{r}_1}{r_{13}^3}$$

$$\frac{1}{2} m_2 v_2^2 = G m_2 m_1 \int \frac{\bar{r}_{21} \cdot d\bar{r}_2}{r_{21}^3} + G m_2 m_3 \int \frac{\bar{r}_{23} \cdot d\bar{r}_2}{r_{23}^3}$$

$$\frac{1}{2} m_3 v_3^2 = G m_3 m_1 \int \frac{\bar{r}_{31} \cdot d\bar{r}_3}{r_{31}^3} + G m_3 m_2 \int \frac{\bar{r}_{32} \cdot d\bar{r}_3}{r_{32}^3}$$

the equations are now added to find

$$\sum \frac{1}{2} m_i v_i^2 = G m_i m_j \int \frac{\bar{r}_{ij} \cdot d\bar{r}_i}{r_{ij}^3} + G m_i m_j \int \frac{\bar{r}_{ij} \cdot d\bar{r}_j}{r_{ji}^3}$$

$$= - G m_i m_j \int \frac{d\bar{r}_{ij}}{r_{ij}^3}$$

$$= G \frac{m_i m_j}{r_{ij}}$$

so that

$$\sum \left(\frac{1}{2} m_i v_i^2 - G \frac{m_i m_j}{r_{ij}} \right) = \text{Const} \tag{13}$$

This is the tenth integral which represents the total conservation of energy. With eighteen integrals necessary for solution and only ten integrals that are defineable further pursuit of an analytic solution of the three-body problem is impossible.

If a restricted three-body system is assumed one further integral may be found to offer new information. This

integral, called the Jacobi integral, does not solve the problem in analytic form however.

The following two assumptions are adopted to formulate a restricted three-body problem:

1) $m_3 \ll m_1$ and m_2

2) m_1 and m_2 are in circular orbits about each other.

With this preliminary introduction to the three-body and the restricted three-body problems concluded attention will now be turned to a singular perturbation solution to the restricted three-body problem formulated by Lagerstrom and Kevorkian [2]. The development will be outlined in notation of Reference 2, however for more detailed discussion and mathematical details the reader is referred to Ref. 2.

C. FORMULATION OF THE RESTRICTED THREE-BODY PROBLEM

With a brief discussion of the general three-body problem completed and with an example problem of matched inner and outer expansions discussed, attention is now turned toward a discussion of Lagerstrom and Kevorkian's work in the application of this expansion solution to a restricted three-body problem in earth-moon space.

The planar restricted three-body problem may be formulated considering the notation used in Figure 5. In this figure x-y is an earth centered frame of reference and $\bar{x}$-$\bar{y}$ is a moon centered frame of reference.

Motion in the x-y system may be considered to be Keplerian about the earth with only small perturbation terms due to the moon until the particle comes into the influence of the moon's gravity field. At this time the perturbation from the moon is no longer small, hence the second characteristic length which results in the singular perturbation problem and allows the use of the boundary layer "stretching" type solutions. The effect of the moon destroys the uniform validity of the Keplerian solution about the earth. If the mass fraction of the earth-moon system is used as a scale factor one may discuss the motion near the moon in terms of the inner or $\bar{x}$-$\bar{y}$ stretched coordinate system.

Considering the outer problem first it is possible to write the planar equations of motion in x-y space directly as:

$$\frac{d^2x}{dt^2} + (1-\mu)\frac{x}{r^3} = \mu f$$

$$\frac{d^2y}{dt^2} + (1-\mu)\frac{y}{r^3} = \mu g \tag{14}$$

where

$$f = \frac{\xi_m - x}{r_m^3} - \xi_m \qquad \text{and} \qquad g = \frac{\eta_m - g}{r_m^3} - \eta_m$$

If the terms on the right hand side of the previous equations are set to zero, one has the Keplerian or two-body equations near the earth. The Keplerian integrals will change little near the earth. These integrals are those

of energy, angular momentum, and position. Recalling the equations for the three-body problem discussed earlier one has,

$$\sum m_i(\bar{r}_i \times \bar{v}_i) = \text{CONST}$$

and

$$\sum \left(\tfrac{1}{2} m_i v_i^2 - \frac{G m_i m_j}{r_{ij}}\right) = \text{CONST}$$

Rewriting these in two-dimensional coordinates in the notation of Lagerstrom one has

$$\frac{1}{2}\left[\dot{x}^2 + \dot{y}^2\right] - \frac{1-\mu}{r} = \text{CONST} = h_e \tag{15}$$

and

$$x\dot{y} - y\dot{x} = \text{CONST} = \ell_e \tag{16}$$

Two equations specifying the position of the orbit in the x-y plane may also be written as

$$\begin{aligned} p_e &= \ell_e \dot{x} + (1-\mu) y/r \\ q_e &= -\ell_e \dot{y} + (1-\mu) x/r \end{aligned} \tag{17}$$

These are written such that if $p_e = 0$ the motion is symmetric about the x-axis. Also if $q_e > 0$ the x-coordinate of the perigee is negative for $\ell_e \neq 0$ and always non-negative for $\ell_e = 0$.

If the initial conditions are specified in terms of these integrals, (which will be done shortly) the motion of the particle in the Keplerian orbit may be determined.

The launch is considered to be at perigee, hence $p_e = 0$ is given.

The angular momentum is defined also to be rv cos ϕ. If perigee is considered $\phi = 0$ and cos $\phi = 1$, therefore, $\ell_e = rv$. If this is non-dimensionalized with respect to the angular momentum of the earth-moon system (moon's period) the non-dimensional angular momentum may be defined as,

$$\lambda = \frac{rv}{\left[\mu D G (m_e + m_m)\right]^{1/2}} \tag{18}$$

The non-dimensional energy is defined in terms of a non-dimensional potential energy and a non-dimensional kinetic energy as,

$$\rho^2 = \left[(1-\mu)\frac{D}{r} - v^2 D/2G(m_e + m_m)\right] \tag{19}$$

Since h_e, ℓ_e, p_e determine q_e the particle initial conditions are specified in terms of h_e, p_e, ℓ_e, and q_e. Writing these in their non-dimensional form,

$h_e = -\rho^2$, $\ell_e = \mu^{1/2}\lambda$, $p_e = 0$, $q_e > 0$ and $t = O(\mu^{3/2})$

one is now able to specify the initial conditions of velocity, altitude, and position in terms of initial Keplerian integrals. The following Keplerian relationships at perigee are known:

$$-x_p = r_p = a(1-e)$$

$$\dot{x} = 0$$

$$\dot{y}^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Replacing h_e by $-\rho^2$, ℓ_e by $\mu^{1/2}\lambda$ and p_e by 0 at $x = 0$ and carrying out some algebraic manipulations the initial

values for the coordinates and velocities become:

$$x = -\frac{1-\mu}{2\rho^2}\left\{1 - \left[1 - \frac{2\mu\rho^2\lambda^2}{(1-\mu)^2}\right]^{1/2}\right\}$$

$$y = 0 \tag{20}$$

$$\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{2\mu^{1/2}\lambda\rho^2}{1-\mu}\left[1-\left(1-\frac{2\mu\rho^2\lambda^2}{(1-\mu)^2}\right)^{1/2}\right]$$

where

$$a = (1-\mu)/2\rho^2$$

and

$$e = \left\{1 - \left[2\mu\rho^2\lambda^2/(1-\mu)^2\right]\right\}^{1/2}$$

It is also beneficial to transform the motion equations to the independent variable x rather than t. One transformation is shown below and the rest follow similar steps to produce the transformed equations. Writing

$$\dot{y} = \frac{dy}{dx}\dot{x} = \frac{dy/dx}{dt/dx}$$

$$\frac{d}{dt}(\dot{y}) = \frac{d\dot{y}}{dx}\frac{dx}{dt} = \frac{d}{dx}\left[\frac{dy/dx}{dt/dx}\right]\frac{1}{t'}$$

$$= \frac{d}{dx}\left[\frac{y'}{t'}\right]\frac{1}{t'}$$

$$= \frac{y''t' - y't''}{(t')^2}\,\frac{1}{t'}$$

finally

$$\frac{d^2y}{dt^2} = \frac{y''t'}{(t')^2} - \frac{y't''}{(t')^3}$$

A similar development may be used to transform d^2x/dt^2, h_e, and ℓ_e so that these equations are finally available in the form:

$$-\frac{t''}{t'^3} + (1-\mu)\frac{y}{r^3} = \mu f$$

$$\frac{y''}{t'^2} - \frac{t''y'}{t'^3} + (1-\mu)\frac{y}{r^3} = \mu q \tag{21}$$

$$\frac{dh_e}{dx} = \frac{d}{dx}\left[\frac{1+y'^2}{2t'^2} - \frac{1-\mu}{r}\right] = \mu\left[f + gy'\right]$$

$$\frac{dl_e}{dx} = \frac{d}{dx}\left[\frac{xy'-y}{t'}\right] = \mu t'(xq - yf)$$

D. OUTER EXPANSION

Attempting a solution of the equations of motion for the particle in earth-moon space in a power series in μ one has;

$$y(x,\mu) = \mu^{1/2} y_{1/2}(x,\mu) + \mu y_1(x) + \cdots \tag{22}$$

$$t(x,\mu) = t_0(x,\mu) + \mu t_1(x) + \cdots \tag{23}$$

The terms $\mu^{\frac{1}{2}} y_{\frac{1}{2}}$ and t_0 are the exact solutions for the equations of motion with the perturbation terms on the right hand side set to zero. In common with celestial mechanics notation, these terms are written as,

$$t_0 = a^{3/2}\left[E(x) - e \sin E(x)\right] \tag{24}$$

$$\mu^{1/2} y_{1/2} = -b \sin E(x) \tag{25}$$

where

$$E(x) = \sin^{-1}\left[1 - \left(e - \frac{x}{a}\right)^2\right]^{1/2}$$

$$b = a(1-e^2)^{1/2}$$

$$a = (1-\mu)/2\rho^2$$

$$e = \left[1 - 2\mu\rho^2\lambda^2/(1-\mu)^2\right]^{1/2}$$

If $t_0(x,\mu)$ and $\mu^{1/2}y_{1/2}(x,\mu)$ are further expanded as

$$t_0(x,\mu) = t_{00}(x) + \mu t_{01}(x) + \cdots \tag{26}$$

$$y_{1/2}(x,\mu) = y_{1/2\,0}(x) + \mu y_{1/2\,1}(x) + \cdots \tag{27}$$

where by appropriate expansions in powers of μ one finds,

$$t_{00}(x) = \left[\sin^{-1} 2\rho(x-\rho^2x^2)^{1/2} - 2\rho(x-\rho^2x^2)^{1/2}\right] / 2^{3/2}\rho^3 \tag{28}$$

$$t_{01}(x) = -t_{00}(x) + \left[2(1-\rho^2\lambda^2)x^2 + 3x\lambda^2\right] / 2\left[2x(1-\rho^2x)\right]^{1/2} \tag{29}$$

$$y_{1/2\,0}(x) = 2^{1/2}\lambda(x-\rho^2x^2)^{1/2} \tag{30}$$

$$y_{1/2\,1}(x) = -\lambda\rho(2x+\lambda^2)(1-2\rho^2x)/2\rho\left[2(x-\rho^2x^2)\right]^{1/2} \tag{31}$$

It is also necessary to insure that the moon and the particle pass within the scale distance μ of each other. The phase constant expansion which includes this condition in the analysis is,

$$T = T_0 + \mu^{1/2} T_{1/2} + \mu T_1 \tag{32}$$

where

$$T_0 = t_{00}(1)$$

$$T_{1/2} = -y_{1/2\,0}(1)$$

and T_1 is an arbitrary term which assures that the moon may be allowed to interact with the particle as desired.

The perturbation terms $t_1(x)$ and $y_1(x)$ may also be determined, the details of which are long and tedious and may be found on pages 689-690 of Shi and Eckstein [6]. The final form of these equations is:

$$t_1(x) = \int_0^x f_o(z)\,[\alpha(x) - \alpha(z)]\,dz \tag{33}$$

$$y_1(x) = x\int_0^x z\,t'_{oo}(z)\,g_o(z)\,[\beta(x) - \beta(z)]\,dz \tag{34}$$

where

$$\alpha(z) = -\left[z^{1/2}(1-\rho^2 z)^{1/2}/2 + z^{1/2}/(1-\rho^2 z)^{1/2} - \left(\tfrac{3}{2\rho}\right)\sin^{-1}\rho z^{1/2}\right]\Big/ z^{1/2}\rho^4 \tag{35}$$

and

$$\beta(z) = -\left\{\left[\frac{1}{\frac{z}{2(1-\rho^2 z)}}\right]^{1/2}\right\} \tag{36}$$

also $f_o(z)$ and $g_o(z)$ are given in terms of the moon's position coordinates as,

$$f_o(z) = \frac{\cos[t_{oo}(z) - T_o] - z}{\{1 + z^2 - 2z\cos[t_{oo}(z) - T_o]\}^{3/2}} - \cos[t_{oo}(z) - T_o] \tag{37}$$

$$g_o(z) = \frac{\sin[t_{oo}(z) - T_o] - z}{\{1 + z^2 - 2z\cos[t_{oo}(z) - T_o]\}^{3/2}} - \sin[t_{oo}(z) - T_o] \tag{38}$$

If $t_1(x)$ and $y_1(x)$ are now expanded asymptotically as $x \to 1$ the following expansions are produced:

$$t_1(x) = (3 - 2\rho^2)^{-3/2}\log(1-x) - \gamma(\rho) + o(1) \tag{39}$$

$$y_1(x) = (3 - 2\rho^2)^{-1/2}\log(1-x) + \delta(\rho) + o(1) \tag{40}$$

where

$$\gamma = \left\{\int_0^1 f_o(z)\left[\alpha(1) - \alpha(z) + \frac{1}{Q(1-z)}\right\}dz + \frac{1}{Q} \tag{41}$$

$$\delta = \int_0^1 \left\{ z t_{oo}'(z) \, q_o(z) \left[\beta(1) - \beta(z) \right] + \frac{1}{Q(1-z)} \right\} dz + \frac{1}{Q} \tag{42}$$

with

$$Q = (1 + U^2)^{3/2} \tag{43}$$

and

$$U = 1 / t_{oo}'(1) \tag{44}$$

With the terms t_o, t_1, y_1, and $\mu^{\frac{1}{2}} y_{\frac{1}{2}}$ now available the outer expansion for t and y is known and is ready to be matched near the moon.

E. INNER EXPANSION

As the vehicle approaches the moon it comes into a hyperbolic encounter with this body. The hyperbolic encounter may be expressed in terms of inner variables. These inner variables are written as,

$$\bar{x} = \frac{x - \cos(t - T)}{\mu}, \quad \bar{y} = \frac{y - \sin(t - T)}{\mu}, \quad \bar{t} = \frac{t - T_0 - \mu \tilde{\tau}}{\mu}$$

where μ is the scale factor for the inner variables and τ is a constant which must be determined.

The inner variable equations of motion are written as:

$$\frac{d^2 \bar{x}}{d\bar{t}^2} + \frac{\bar{x}}{\bar{r}^3} = - \frac{\mu(1-\mu)(\mu \bar{x} + \cos\psi)}{\bar{r}_e^3} + \mu(1-\mu)\cos\psi \tag{45}$$

$$\frac{d^2 \bar{y}}{d\bar{t}^2} + \frac{\bar{y}}{\bar{r}^3} = - \frac{\mu(1-\mu)(\mu \bar{y} + \sin\psi)}{r_e^3} + \mu(1-\mu)\sin\psi \tag{46}$$

where

$$\bar{r}^2 = \bar{x}^2 + \bar{y}^2$$

$$\bar{r}_c^2 = 1 + 2\mu(\bar{x}\cos\psi + \bar{y}\sin\psi) + \mu^2\bar{r}^2$$

$$\psi = \mu(\bar{t} - \tau) + T_0 - T$$

In the hyperbolic encounter one may express the Keplerian solution in parametric form as,

$$\bar{x} = \bar{a}(\bar{e} - \cosh u)\cos\bar{\theta} \mp \bar{a}(\bar{e}^2 - 1)^{1/2}\sinh u \sin\bar{\theta} \tag{47}$$

$$\bar{y} = \bar{a}(\bar{e} - \cosh u)\sin\bar{\theta} \pm \bar{a}(\bar{e}^2 - 1)^{1/2}\sinh u \cos\bar{\theta} \tag{48}$$

$$\bar{t} = \bar{a}^{3/2}(\bar{e}\sinh u - u) \tag{49}$$

The Keplerian integrals previously discussed expressed for inner variables are

$$\begin{aligned} \bar{h} &= \tfrac{1}{2}\left[\left(\frac{d\bar{x}}{d\bar{t}}\right)^2 + \left(\frac{d\bar{y}}{d\bar{t}}\right)^2\right] - \frac{1}{\bar{r}} \\ \bar{l} &= \bar{x}\frac{d\bar{y}}{d\bar{t}} - \bar{y}\frac{d\bar{x}}{d\bar{t}} \\ \bar{p} &= \bar{l}\frac{d\bar{x}}{d\bar{t}} + \bar{y}/\bar{r} \\ \bar{q} &= -\bar{l}\frac{d\bar{y}}{d\bar{t}} + \bar{x}/r \end{aligned} \tag{50}$$

These integrals are related to the hyperbolic elements by,

$$\bar{a} = 1/2\bar{h}, \quad \bar{e} = \sqrt{1 + 2\bar{h}\bar{l}^2}$$

$$\bar{p} = -\bar{e}\sin\bar{\theta}, \quad \bar{q} = -\bar{e}\cos\bar{\theta}$$

To carry out the matching of the inner and outer expansions the values of $\bar{y}$ and $\bar{t}$ as functions of $\bar{x}$ along the approach asymptote must be determined. These functional

values may be determined by allowing $u \to \infty$. As $u \to \infty$ the following functional forms of $\bar{y}$ and $\bar{t}$ are determined to be:

$$\bar{y} = \frac{V_1}{U_1}\bar{x} - \frac{AV_1 - BU_1}{U_1} + O(\bar{x}^{-1}) \qquad (51)$$

$$\bar{t} = \frac{\bar{x} - A}{U_1} + \bar{a}^{3/2} \log \frac{-2\bar{x}}{U_1 \bar{a}^{3/2} \bar{e}} + O(\bar{x}^{-1}) \qquad (52)$$

The U_1 and V_1 terms are the velocity components on the approach asymptote.

$$U_1 = \frac{2h\bar{p}\bar{l} - (2h)^{1/2}\bar{q}}{1 + 2h\bar{l}^2}$$

$$V_1 = \frac{-2h\bar{q}\bar{l} + (2h)^{1/2}\bar{p}}{1 + 2h\bar{l}^2}$$

where

$$U_1^2 + V_1^2 = 2h$$

and A and B are,

$$A = -\bar{q}/2h \qquad \text{and} \qquad B = -\bar{p}/2h$$

If the negative $\bar{y}$-intercept is considered to be K_1, that is to say $\bar{y} = -K_1$ for $x = 0$,

$$K_1 = \frac{AV_1 - BU_1}{U_1} \qquad (53)$$

and

$$\bar{l} = K_1 U_1 \qquad (54)$$

U_{11} and V_{11}, the velocity components along the departure hyperbola, may be written also

$$U_{11} = \frac{2h\bar{p}\bar{l} + (2h)^{1/2}\bar{q}}{1 + 2h\bar{l}^2}$$

$$V_{11} = \frac{-2h\bar{p}\bar{l} + (2h)^{1/2}\bar{p}}{1 + 2h\bar{l}^2}$$

F. MATCHING OF THE INNER AND OUTER SOLUTIONS

The matching of the inner and outer expansions must be carried out at this time. Returning to the example discussed, the principle behind this matching may be recalled. The only difference in the two problems is that in the present case, since only the first term of the moon inner variable expansion is needed, it is only necessary to rewrite the outer expansion in inner variables and then carry out the matching.

For the particle near the moon the x-coordinate is ξ_m. This occurs when $t = T_0$. If the x-coordinate is specified to be

$$x = 1 - \frac{\mu}{2} T_{1/2}^2$$

this implies that

$$\xi_m = 1 - \frac{\mu}{2} T_{1/2}^2$$

The phase relation was written as,

$$T = T_0 + \mu^{1/2} T_{1/2} + \mu T_1 = T_0 + \mu^{1/2} T_{1/2} + O(\mu) \tag{55}$$

but with $t = T_0$ the equation may be written as,

$$t - T = \mu^{1/2} T_{1/2} + O(\mu)$$

so that

$$\bar{x} = \bar{\bar{x}} + O(1)$$

with

$$\bar{x} = \frac{x - \xi_m}{\mu} = \frac{x - 1 + \frac{\mu}{2} T_{1/2}^2}{\mu}$$

therefore

$$\bar{\bar{x}} = \frac{x - 1 + {}^1\!/_2 T_{1/2}^2}{\mu}$$

If the outer expansion is written as

$$t = t_{00}(x) + \mu[t_{01}(x) + t_1(x)] + \cdots \tag{56}$$

$$y = \mu^{1/2} y_{1/2}(x, 0) + \mu y_1(x) + \cdots \tag{57}$$

and replacing

$$x \quad \text{by} \quad x = 1 + \mu(\bar{x} - T_{1/2}^2/2)$$

$$t = t_{00}(1) + \mu\left[(\bar{x} - \frac{T_{1/2}^2}{2})U^{-1} + t_{01}(1) + (1+U^2)^{-3/2} \log \mu(-\bar{x})\right.$$
$$\left. + Y(\rho)\right] + o(\mu)$$

$$t_{00}(1) = T_0 \qquad \text{AND} \qquad \bar{t} = \frac{t - T_0 - \mu\tau}{\mu}$$

$$\bar{t} = (\bar{x} - \frac{T_{1/2}^2}{2})U^{-1} + t_{01}(1) + (1+U^2)^{-3/2}\{\log[\mu] + \log[-\bar{x}]\}$$
$$+ Y(\rho) - \tau + o(1) \tag{58}$$

also it is recalled that 52 was written

$$\bar{t} = \frac{\bar{x} - A}{U_1} + \bar{a}^{3/2} \log \frac{-2\bar{x}}{U_1 \bar{a}^{3/2} \bar{e}} + o(\bar{x}^{-1})$$

The term $(1 + U^2)^{-3/2} \log \mu$ does not occur in the later equation. It must therefore be cancelled by some other term. Since the only arbitrary term in the inner expansion for $\bar{t}$ is τ, a term in τ must cancel the log μ term, hence,

$$\tau = (1+U^2)^{-3/2} \log \mu + \tau_1 \tag{59}$$

and

$$\bar{t} = U^{-1}\bar{x} + (1+U^2)^{-3/2} \log(-\bar{x}) + t_{01}(1) - \frac{T_{1/2}^2}{2U} + Y(\rho) - \tau_1 + o(1) \tag{60}$$

$\bar{y}$ may likewise be determined to be,

$$\bar{y} = \frac{T_{1/2}^2}{2U} - t_{01}(1) + T_1 + \delta(\rho) - \gamma(\rho) - \frac{\bar{x}}{U} + o(1) \tag{61}$$

Comparison of 51 to 61 and 52 to 60 shows

$$U_1 = U$$

$$V_1 = -1$$

If the equations are written such that,

$$\bar{y} = \frac{V_1}{U_1}\bar{x} - K_1 + o(\bar{x}^{-1}) \tag{62}$$

and

$$\bar{y} = -K - \frac{\bar{x}}{U} + o(1) \tag{63}$$

where

$$-K = \frac{T_{1/2}^2}{2U} - t_{01}(1) + T_1 + \delta(\rho) - \gamma(\rho) \tag{64}$$

$$K_1 = (AV_1 - BU_1)/U_1 \tag{65}$$

Then it is evident that

$$K_1 = K$$

Rewriting $\bar{t}$ as

$$\bar{t} = \frac{\bar{x}}{U} + (1+U^2)^{-3/2}\log(-\bar{x}) + L + o(1) \tag{66}$$

where

$$L = t_{01}(1) - \frac{T_{1/2}^2}{2U} + \gamma(\rho) - \bar{\tau}_1$$

and

$$\bar{t} = \frac{\bar{x} - A}{U_1} + \bar{a}^{3/2}\log\frac{-2\bar{x}}{U_1\bar{a}^{3/2}\bar{e}} + o(x^{-1}) \tag{67}$$

then

$$L = -\frac{A}{U_1} + \bar{a}^{3/2} \log \frac{2}{U_1 \bar{a}^{3/2} \bar{e}}$$

which completes the equation matching process and determines the unknown constants.

These values are used in equations 50 to obtain from substitution

$$\begin{aligned} 2h &= 1 + U^2 \\ \bar{l} &= KU \\ \bar{p} &= KU^2 + (1+U^2)^{-1/2} \\ \bar{g} &= KU - U(1+U^2)^{-1/2} \end{aligned} \tag{68}$$

The term τ_1 may also be completely determined to be

$$\tau_1 = t_{01}(1) - \frac{T_{1/2}^2}{2U} + Y(p) + \frac{A}{U} - \bar{a}^{3/2} \log \frac{2}{\bar{a}^{3/2} U \bar{e}} \tag{69}$$

The total or composite expansion solution is the outer expansion plus the inner expansion minus the common part. It may be written as;

$$y = \mu^{1/2} y_{1/2}(x,\mu) + \mu y_1(x) + \mu[G(\bar{x}) - \alpha^*(\bar{x})] + o(\mu) \tag{70}$$

$$t = t_0(x,\mu) + \mu t_1(x) + \mu[E(\bar{x}) - \beta^*(\bar{x})] + o(\mu) \tag{71}$$

where

$$\begin{aligned} \alpha^* &= \frac{V_1}{U_1}\bar{x} - \bar{a}\,\bar{e}(V_1 \cos\bar{\theta} - U_1 \sin\bar{\theta})/U_1 \\ \beta^* &= (\bar{x} - \bar{a}\,\bar{e} \cos\bar{\theta})/U_1 \end{aligned} \tag{72}$$

The solution to first order after moon passage may be obtained by a similar matching procedure as that used above.

Before leaving the analytic theory it is wise to note that this procedure of inner and outer expansions may be

carried out to higher orders of μ which increases the accuracy as the value of the parameter μ increases.

III. THE COMPUTER PROGRAM

The computer program used in this paper was written for the IBM 360 system at the Naval Postgraduate School Computer Facility. It has been run successfully on the OS/CMS time share remote terminal and the standard batch processor unit.

The program is a straight forward computation with the use of Scientific Subroutine Package programs from the sub-program library. These programs are used to evaluate the integrals which appear in the solution near the moon.

All computation is carried out in double-precision notation to maintain as much accuracy as possible. The integration routine is a double-precision sixteen point Gaussian quadrature from the SSP library in the Fortran IV language.

The necessary inputs to the program are the non-dimensional boundary conditions discussed in this paper. From these parameters the program calculates the necessary constants, then the integrals of the motion equations are computed for $x = 1$ (near the moon). With these values the program evaluates the motion in the vicinity of the moon and prints the necessary parameters. Control is then returned to the input mode and a new series of calculations may be carried out. If ρ becomes too large the machine proceeds to stop and ends processing.

The necessary equations, a test case, and the computer program are contained in the Appendix. A flow chart may be found in Figures 17 and 18. All equations are explained in References 7, 8, and 9 and the same notation is used as much as possible.

IV. DISCUSSION OF RESULTS

A. VERIFICATION OF COMPUTER RESULTS AGAINST THOSE OF REFERENCE 7

A computer program to solve the restricted earth-moon three-body problem was written. The first test of this program was against the data presented in Reference 7. Figures 6, 7, and 8 show the agreement of the present results with those of Ref. 7. The results for γ and δ also agree with Table 1 of Reference 7. As a final check examples one through nine of Reference 7 were tested and comparison of results was good.

No data of any form for values of ρ greater than 0.707 could be made to agree with the published data of Reference 7. The inability of the program to predict accurate results for this case of ρ greater than 0.707 was unresolved. The probable cause of the instability of the solution is due to the denominator of the $f_0(x)$ term rapidly vanishing near this point. The equation for $f_0(x)$ was rewritten in several forms, all to no avail. The equation was also expanded in a Taylor's series about 0. This expansion gave some improvement in the prediction of γ and δ, but the improvement was not sufficient to be used for numerical work.

B. PARAMETRIC STUDY IN THREE DIMENSIONAL EARTH-MOON TRAJECTORIES

The program used to predict the results of Reference 7 was modified to include three dimensional orbit consideration

for values of $i = O(1)$. The modification still led to the same results for planar orbits and a study for inclined orbits was carried out. Cases were run to predict the effect of variations of energy, angular momentum, position of the moon and orbital inclination on the space vehicle as it approached the moon's position.

The present program predicts all of the important parameters near the moon. The effect on changes in initial conditions can be observed in the changes in these near moon parameters. One of the most important of these parameters and one that requires knowledge of the other parameters is the parameter of closest approach to the moon. This parameter has been chosen to show the effects of varying initial conditions on the moon encounter.

Figures 9 and 10 show how the initial launch conditions are related to altitude and velocity at launch from perigee. The altitude limit on perigee shown in Figure 9 indicates values of lambda must be less than 3.0. Figure 10 shows the velocity at perigee. Since conventional launch vehicles cannot attain values of velocity much greater than around 40,000 kms/hr an outside limit on velocity was taken to be 50,000 kms/hr. This, then implies that lambda should be greater than 1.0. For a given launch condition the values of rho and lambda can be related to the velocity and altitude at perigee. The perigee launch gives the angle of perigee to be π and time of launch to be $t = 0$. Hence, this defines the launch conditions completely.

Figure 11 shows the effect of changes in the initial energy, rho, relative the earth on the distance of closest approach. It is apparent that as the energy is reduced (rho becomes greater) the distance of closest approach to the moon is increased. The effect of orbit inclination on the distance of closest approach is also evident. This Figure predicts an increase in the distance as the inclination angle of the orbit is increased.

Figure 12 depicts the effects of changes in the angular momentum, lambda, relative to the earth on the approach distance to the moon. As the angular momentum is increased (this corresponds to an increase in velocity and/or perigee height) the distance of closest approach to the moon is increased. The effect of the inclination of the orbit shows up in this Figure as before to indicate greater approach distances for higher values of inclination.

Figure 13 can be used to predict two effects which occur due to parameter changes. The effect of T_1 on the closest approach distance is due to the variation of the moon's position at launch of the vehicle. For positive values of T_1 the moon and the vehicle interact very closely, whereas for negative values of T_1 the moon passes far from the vehicle.

The effect of a shift on the distance of closest approach may also be seen in Figure 13. Smaller values of the apse angle allow for closer interaction with the moon.

The value for planar orbits is independent of the apse angle and therefore remains the same.

Figures 16, 17, and 18 show the effect of these same parameters on the distance of closest approach for orbits of only slight inclinations. As can be seen from these figures the effects of changes in rho, lambda, and T_1 are the same as before. As would be expected changes in the apse angle and small changes in the inclination angle from the planar case cause no significant change in the distance of closest approach as predicted in the planar case.

Figures 14 and 15 show the three-dimensional geometry of the earth orbit and the moon passage. The inclination angle, i, discussed is the angle between the earth-moon plane and the orbit plane. The apse angle, ω, is the angle from the node line, Ω, to the apse axis. Similar parameters are shown for the moon passage orbit.

V. CONCLUSIONS AND RECOMMENDATIONS

The method of uniformly valid asymptotic expansions appears to have merit as an initial analytic tool. The ability of this solution to predict values of parameters of earth-moon trajectories with such a small expenditure of computer time well justifies use of these expansion solutions. A quick parametric variational study could predict "ball-park" initial conditions which would greatly reduce the time of total solution of a trajectory problem.

Observing the results of the analysis of this paper it appears that one might conclude that the closest approach possible occurs when the launch energy is as high as possible. The angular momentum relative to the earth must be low also for this closest approach to occur. The moon's position which is fixed by T_1 and ω must be such that it is just ahead of the particle as they come near each other.

The only other parameter which must be considered is the launch inclination. The minimum value of closest approach is indicated to occur for planar trajectories.

From this discussion one then concludes that minimum distance of closest approach occurs for a planar launch at the highest value of energy and lowest value of angular momentum possible. These are attained for near earth launch with high perigee velocities.

It is recommended that further study should be conducted in the vicinity of the minimum energy solution region. This is the region of the previous mentioned instability and is very important in the study of low energy transfers where most of todays travel takes place.

It would also be interesting to release some of the restrictions on the analysis in order to apply the solution to interplanetary problems.

Kevorkian [8] has extended the analysis to higher order terms and a comparison if this result with those higher order solutions would also be of interest.

Additional work is necessary before the present analysis can be applied to a full spectrum of trajectory problems. The present work however indicates that useful information may be gained from this form of solution for the restricted three-body problem and that this solution has merit as a potential analytical tool for initial design studies.

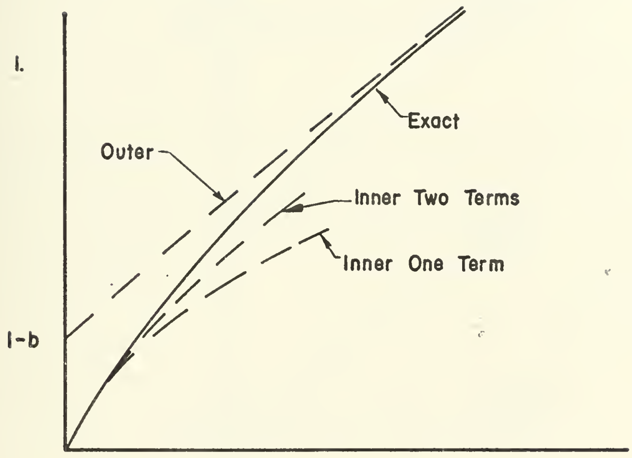

FIGURE 1. AMES, VAN DYKE BOUNDARY LAYER PROBLEM.

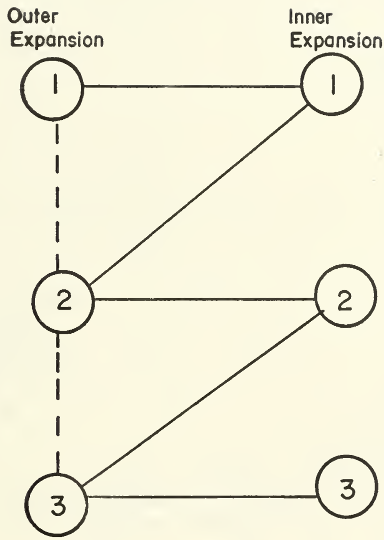

FIGURE 2. FLOW CHART OF MATCHING PRINCIPLE.

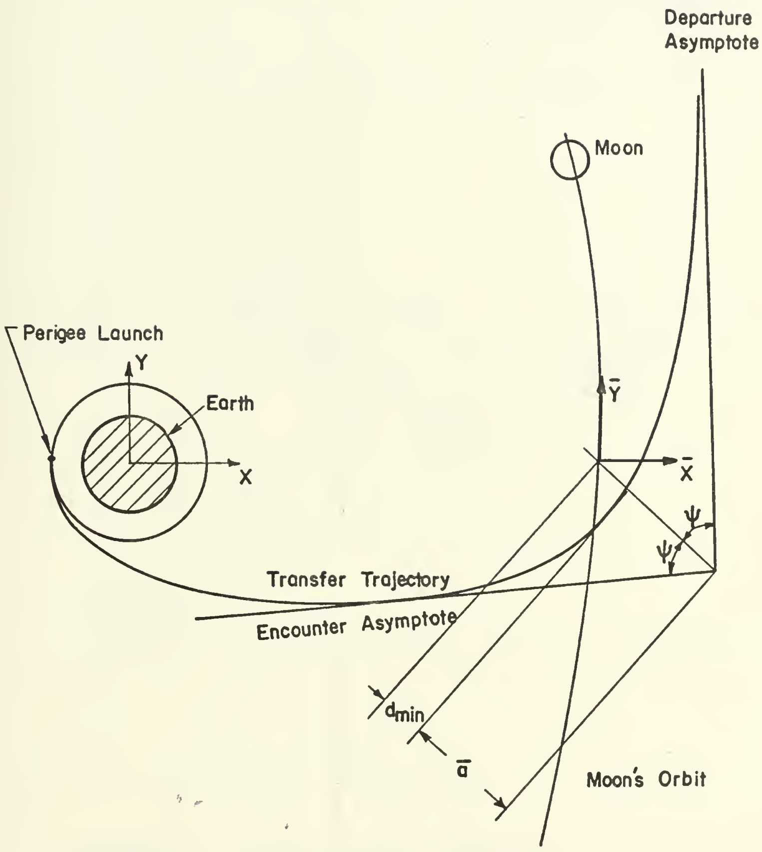

FIGURE 3. EXAMPLE EARTH-MOON HYPERBOLIC ENCOUNTER TRAJECTORY.

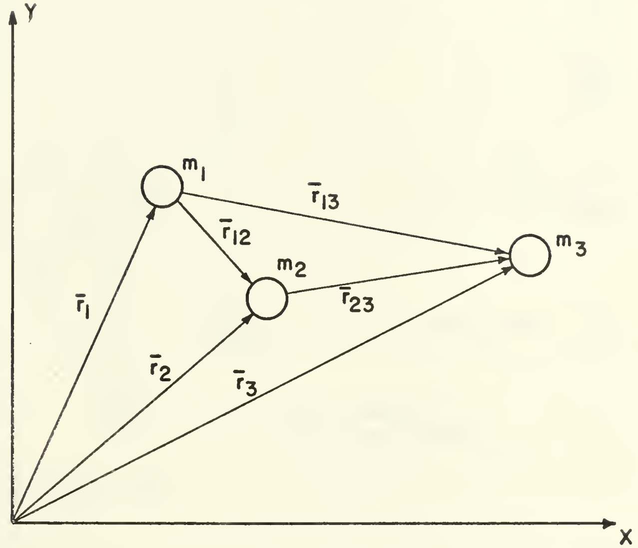

FIGURE 4, THREE - BODY GEOMETRY

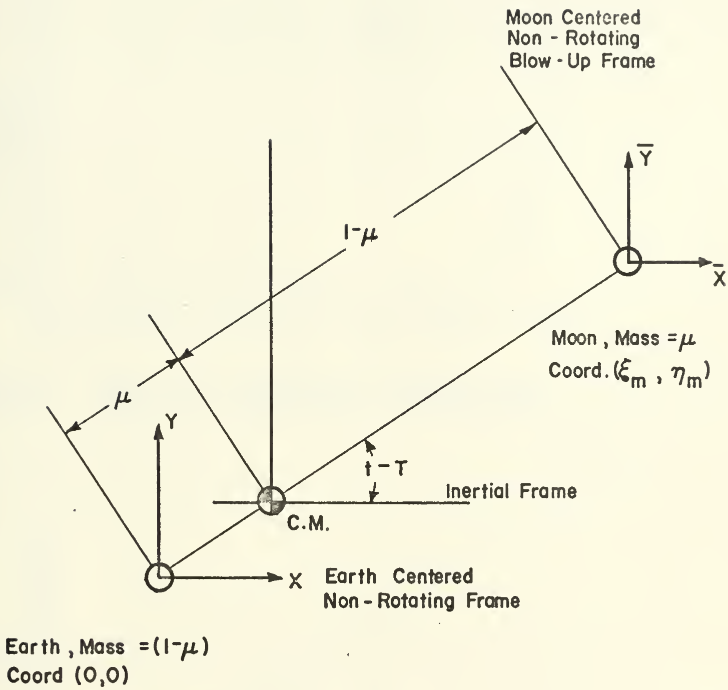

FIGURE 5 , EARTH-MOON COORDINATE SYSTEMS FOR THE RESTRICTED 3-BODY PROBLEM.

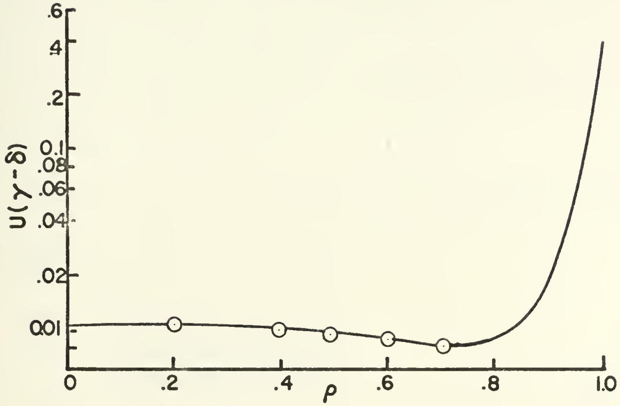

FIGURE 6. ANGULAR MOMENTUM CORRECTION.

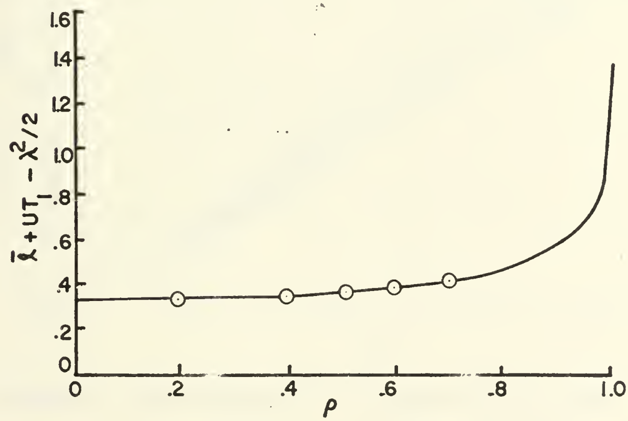

FIGURE 7. ANGULAR MOMENTUM RELATIVE TO THE MOON.

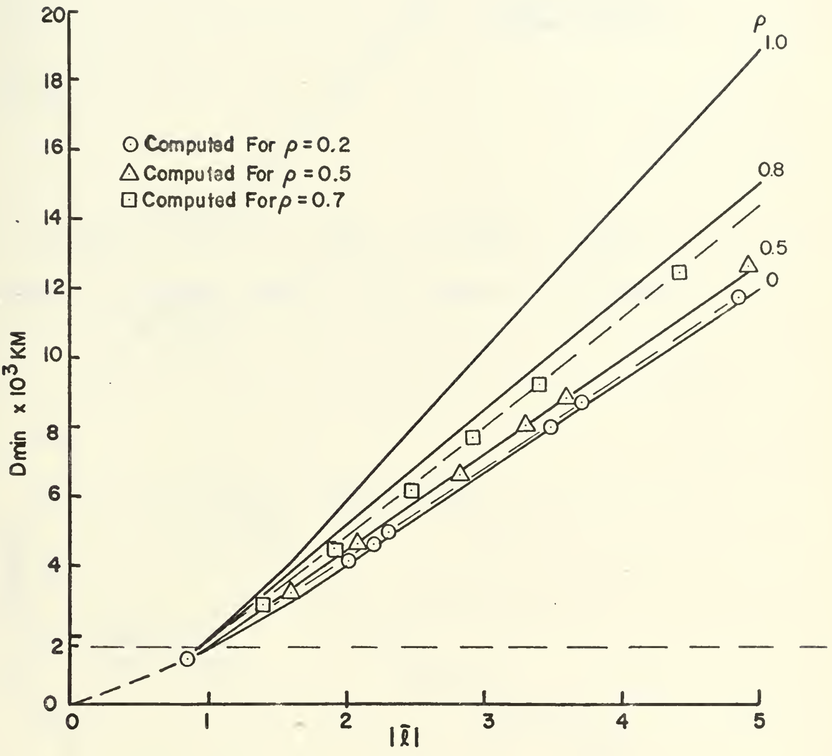

FIGURE 8. DISTANCES OF CLOSEST ENCOUNTER WITH MOON.

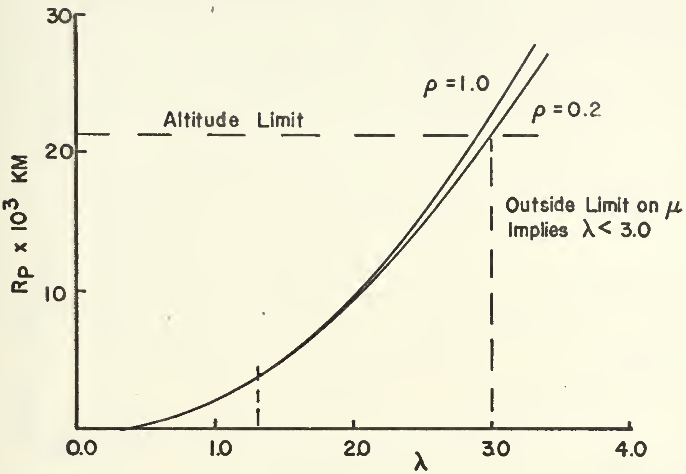

FIGURE 9. PERIGEE HEIGHT VS. LAMBDA FOR VALUES OF RHO.

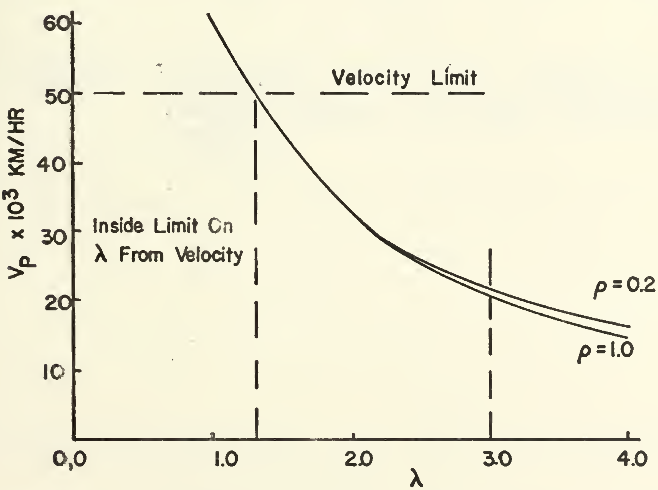

FIGURE 10. PERIGEE VELOCITY VS. LAMBDA FOR VALUES OF RHO.

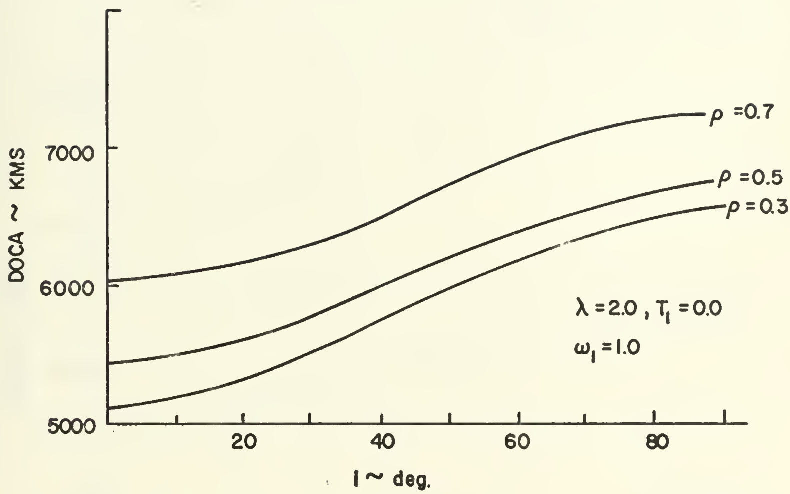

FIGURE 11. INCLINATION VS. CLOSEST APPROACH FOR VARYING ENERGY, ALL OTHER PARAMETERS CONSTANT.

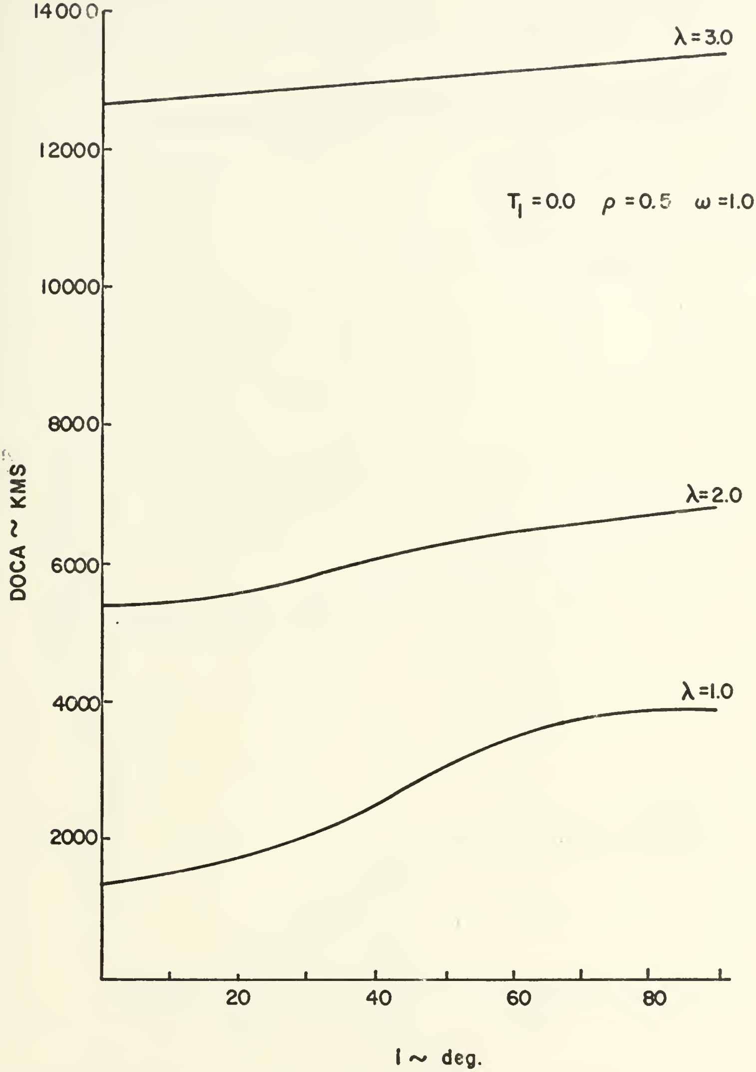

FIGURE 12. INCLINATION VS. CLOSEST APPROACH FOR VARYING ANGULAR MOMENTUM, ALL OTHER PARAMETERS CONSTANT.

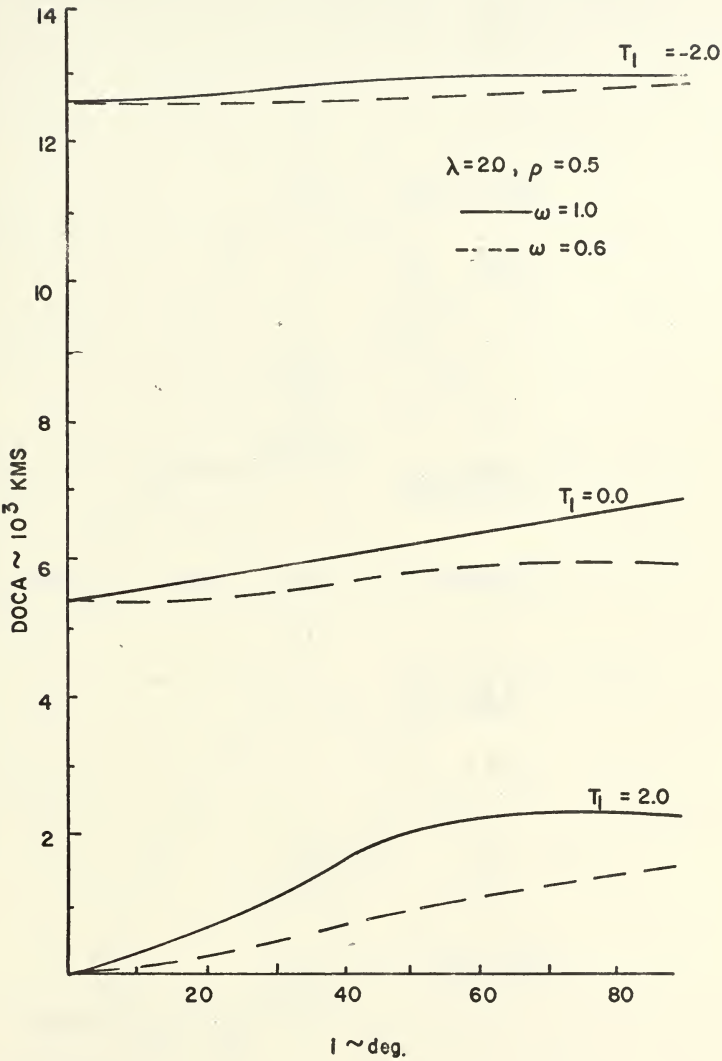

FIGURE 13. INCLINATION VS. CLOSEST APPROACH FOR VARYING PHASE CONSTANT (T_1) ALL OTHER PARAMETERS CONSTANT.

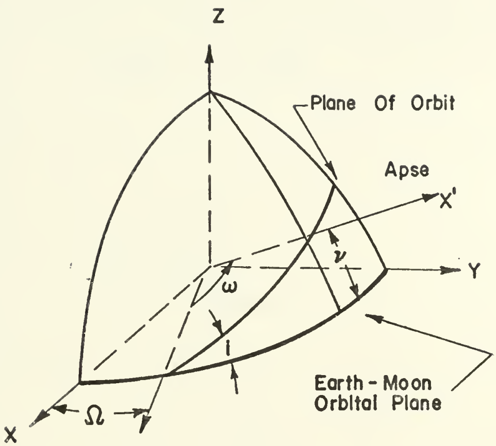

FIGURE 14. EARTH ORBIT 3-D GEOMETRY

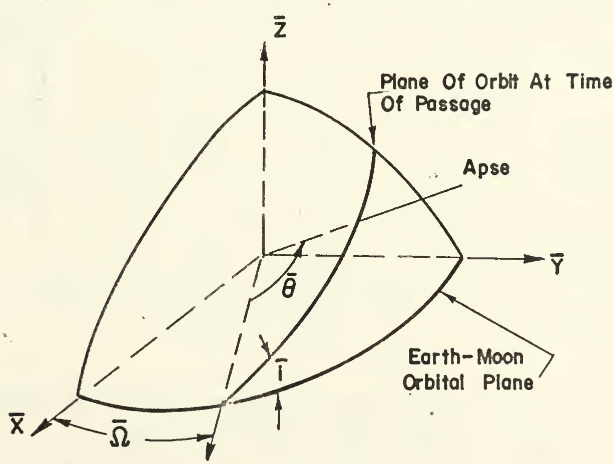

FIGURE 15. MOON PASSAGE 3-D GEOMETRY

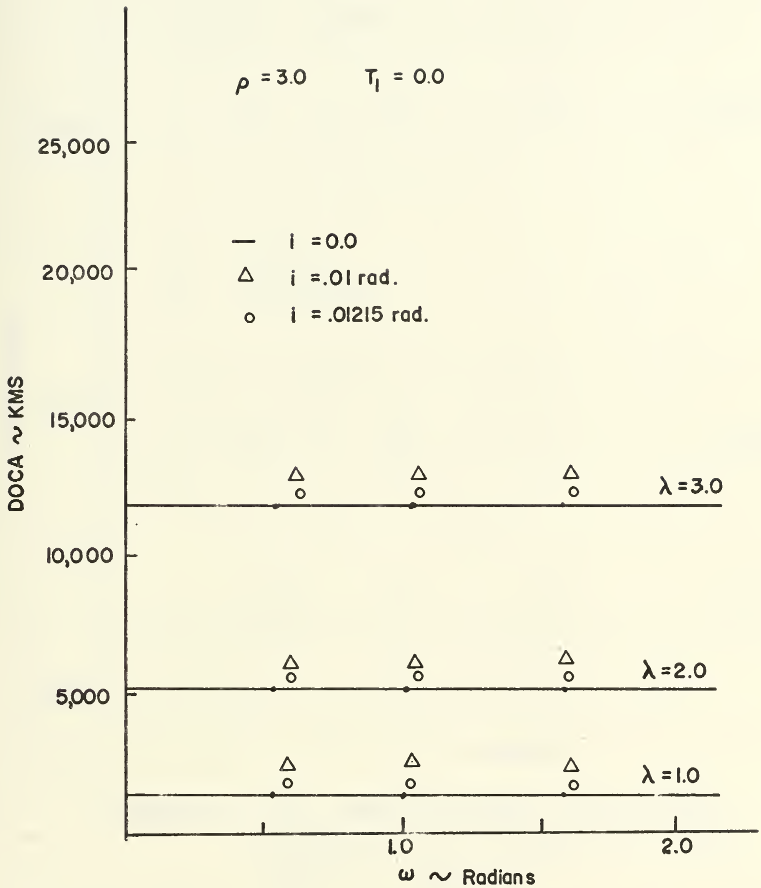

FIGURE 16. APSE ROTATION ANGLE VS. CLOSEST APPROACH FOR 3 VALUES OF i WITH ρ AND T_1 CONSTANT

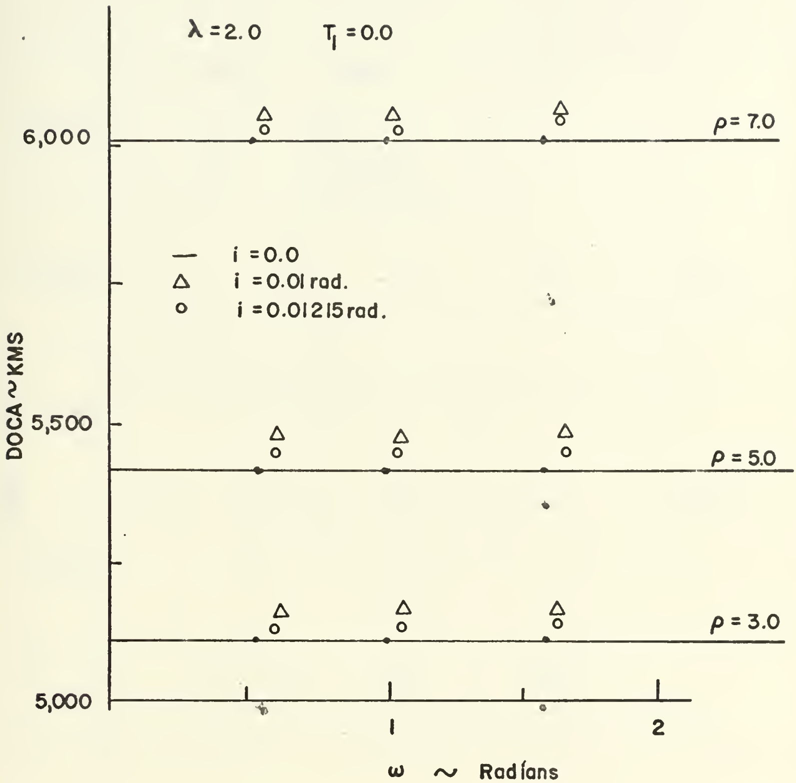

FIGURE 17. APSE ROTATION ANGLE VS. CLOSEST APPROACH FOR 3 VALUES OF i WITH λ AND T_1 FIXED

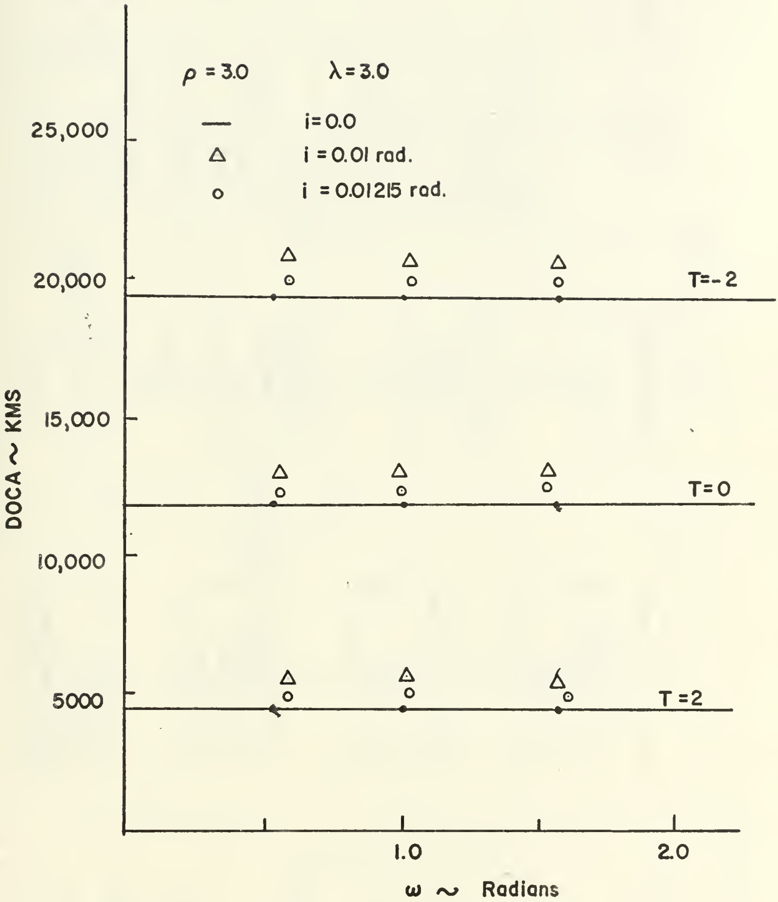

FIGURE 18. APSE ROTATION ANGLE VS. CLOSEST APPROACH FOR 3 VALUES OF i WITH ρ AND λ CONSTANT.

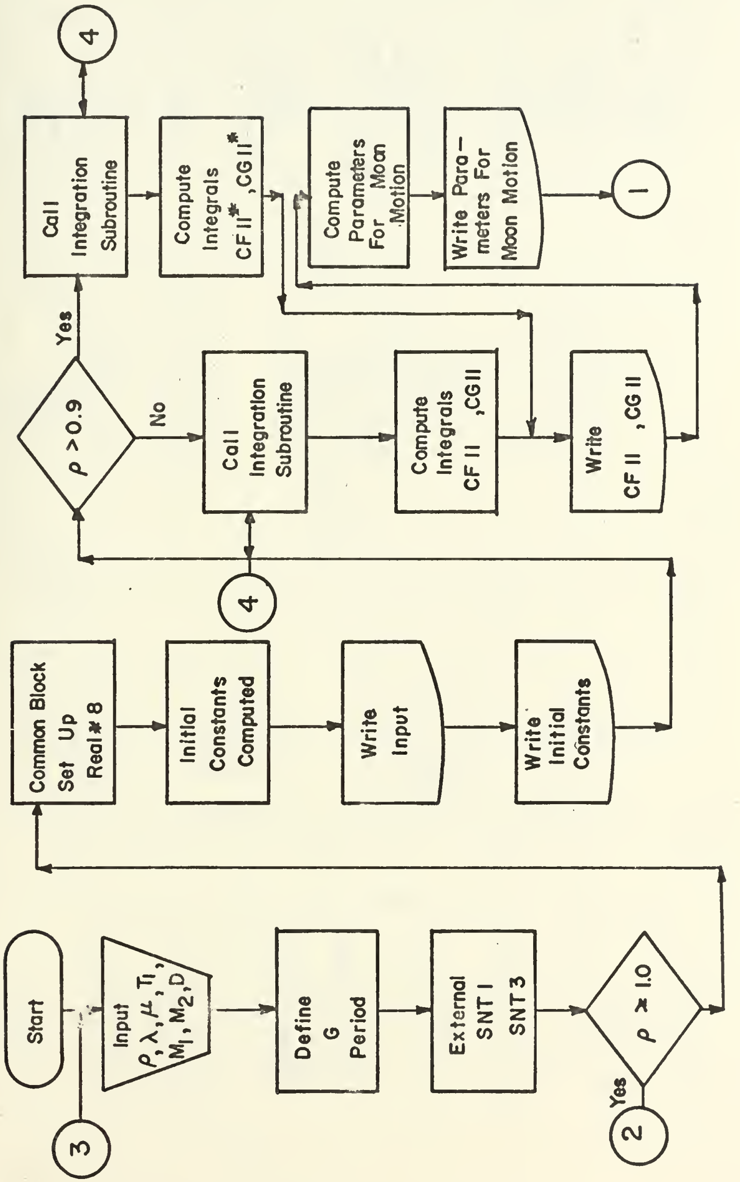

FIGURE 19. PROGRAM FLOW CHART PART A

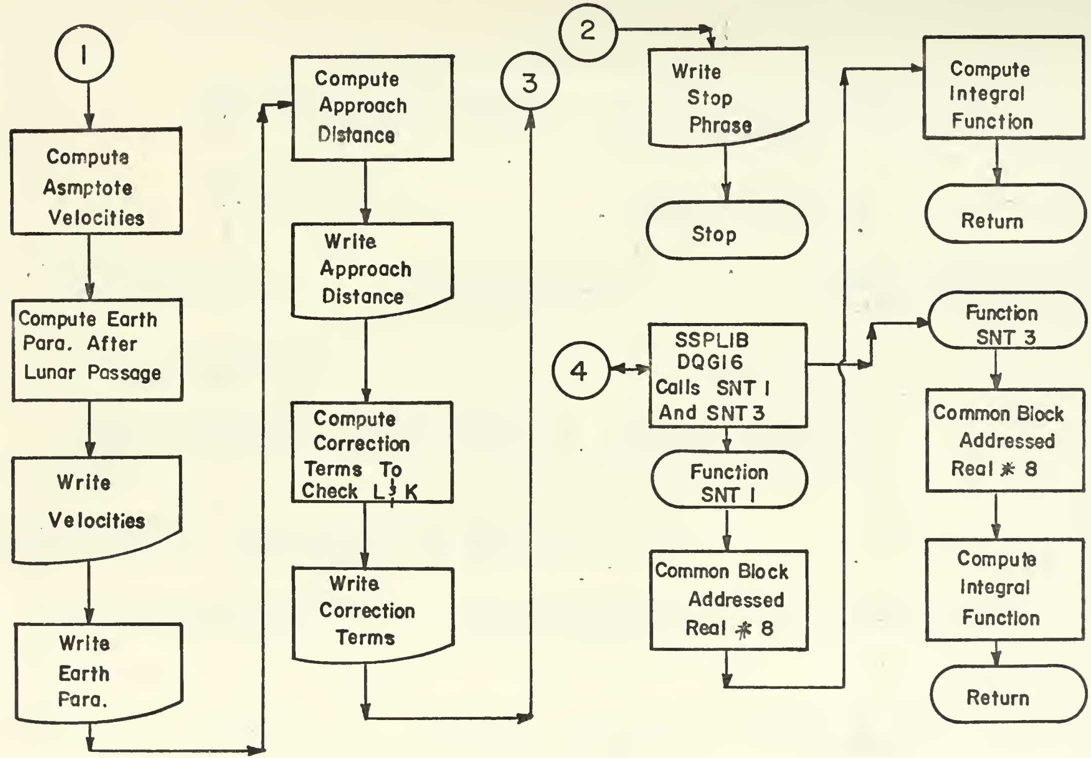

FIGURE 20 PROGRAM FLOW CHART PART B

```
                    THE INITIAL PARAMETERS ARE

          MASS PARENT =0.597740D 25KGS  MASS TARGET=0.734550D 23KGS
          SEP DIS=0.384400D 06KMS       PERIOD=0.273144D 02DAYS
          GRAVITY CONSTANT=0.496720D-09KMS**3/DAY**2*KG

          RHO=0.500000D 00                 MU=0.121500D-01
           LAMBDA=0.200000D 01             T1=0.0
           I=0.0                           WSUB1=0.100000D 01

     TOOP1=0.816497D 00     TOO1=0.512432D 00          Y1201=0.244949D 01
     CAPTO=0.512432D 00     CAPT12=0.244949D 01        CAPT=0.782432D 00
     U=0.122474D 01         Q=0.395285D 01             ALPHA1=-.191393D 00

          ////////// THE INTEGRALS ARE //////////

      GAM= 0.46638285D 00                    DEL= 0.45863350D 00
          ****TARGET ORBIT PARAMETERS****

      HBAR=0.125000D 01   LBAR=0.238189D 01   ABAR=0.400000D 00
      PBAR=0.354967D 01   QBAR=0.160730D 01   EPSBAR=0.389650D 01
      THEBAR=-.751296D 02 TAUBAR=0.635049D 00OMEGAB=-.684719D 00
     IBAR=0.0

       ***** INCOMING AND OUTGOING VELOCITIES *****
   UI=0.122474D 01   VI=-.100000D 01  WI=0.0
   UII=0.127169D 01  VII=-.613200D 00 WII=0.0

     PSIDEGREE=0.169178D 02        HEII=0.136800D 00   LEII=0.386800D 00
     PEII=0.491889D 00             QEII=0.123719D 01

  CLOSEST APPROACH = 0.54113902D 04

 U(GAMMA-DELTA)= 0.94909746D-02    LBAR+UT1-(LAMB**2)/2= 0.38189225D 00

 <<<<<<<<<<<<ALL FOR THIS CALCULATION >>>>>>>>>>>>>>

 COMPUTER PROGRAM TEST CASE
```

LIST OF EQUATIONS USED IN THE IBM SOLUTION

The flowchart of Figures 17 and 18 shows how the logic of the computer solution is carried out. The following equations are those used in the solution on the computer.

INPUT

$$\rho = \left\{ \frac{(1-\mu)D}{r_p} - \frac{V_p^2 D}{2G(m_e+m_m)} \right\}^{1/2}$$

$$\lambda = \frac{r_p V_p}{\left\{ \mu D G(m_m+m_e) \right\}^{1/2}}$$

$$\mu = 0.01215 \doteq \frac{m_m}{m_e+m_m}$$

$T_1 =$ Phase Constant

$G = 4.967 \times 10^{-10}$ KM^3/KG-DAY^2

$$PER = 2\pi \sqrt{\frac{D^3}{2G(m_e+m_m)}} \doteq 27.3217 \text{ DAYS}$$

$I =$ ORBIT INCIDENCE ANGLE

$\omega =$ APSE LOCATION

EVALUATION OF TERMS FOR X = 1

$$T001 \equiv t_{00}(1) = \left[\sin^{-1} 2\rho(1-\rho^2)^{1/2} - 2\rho(1-\rho^2)^{1/2} \right] / 2^{3/2}\rho^3$$

$$TOOP1 \equiv t'_{00}(1) = \left[1/2(1-\rho^2) \right]^{1/2}$$

$$Y1201 \equiv y_{12}(1) = 2^{1/2}\lambda(1-\rho^2)^{1/2}$$

$$CAPTO \equiv T_0 = t_{00}(1)$$

$$CAPT12 \equiv T_{1/2} = \lambda U \cos i$$

$$CAPT \equiv T = T_0 + \mu^{1/2} T_{1/2} + \mu T_1$$

$$ALPHA1 \equiv \alpha_1 = -\left[(1-\rho^2)^{1/2}/2 + 1/(1-\rho^2)^{1/2} - \frac{3}{2\rho} \sin^{-1}\rho \right] / 2^{1/2}\rho^4$$

$$U \equiv U = 1/t'_{00}(1)$$

$$Q \equiv Q = (1+U^2)^{3/2}$$

EVALUATION OF TERMS IN $\bar{x}$-$\bar{y}$ (MOON) SPACE

$$\text{HBAR} \equiv \bar{h} = (1+U^2)/2$$

$$\text{XLBAR} \equiv \bar{l} = \left[1 + \lambda^2(1+U^2\sin^2 i)/2 + U(\gamma-\delta) - U(T_0+T_1) - \xi(1,0)\right]/\cos\bar{i}$$

$$\text{ABAR} \equiv \bar{a} = 1/2\bar{h}$$

$$\text{PBAR} \equiv \bar{p} = U\bar{l} + (2\bar{h})^{-1/2}$$

$$\text{QBAR} \equiv \bar{q} = \bar{l} - U(2\bar{h})^{-1/2}$$

$$\text{EPSBAR} \equiv \bar{e} = \{1 + 2\bar{h}\bar{l}^2\}^{1/2}$$

$$\text{TAUBAR} \equiv \bar{\tau} = \bar{a}^{3/2}\log(\mu\bar{a}^{3/2}\bar{e}\tfrac{U}{2}) + [(1-\bar{a})/U]\bar{l}\cos\bar{i} + \bar{a}^{3/2} + T_1 + \delta$$

$$\text{THEBAR} \equiv \bar{\theta} = \tan^{-1}(-\bar{l}(1+U^2)^{1/2})$$

$$\text{IBAR} \equiv \bar{i} = \sin^{-1}(-\xi(1+U^2)^{1/2}/\bar{l}$$

$$\text{OMEGB} \equiv \bar{\Omega} = \tan^{-1}(-1/U)$$

EVALUATION OF VELOCITY COMPONENTS

$$\text{U1} \equiv U_1 = U$$

$$\text{V1} \equiv V_1 = -1$$

$$\text{W1} \equiv W_1 = 0$$

$$\text{U11} \equiv U_{11} = (1/\bar{e}^2)(1+U^2)^{1/2}[(1+U^2)\bar{l}^2-1]\cos\bar{\Omega} - (\tfrac{2}{\bar{e}^2})(1+U^2)\sin\bar{\Omega}\cos\bar{i}$$

$$\text{V11} \equiv V_{11} = (1/\bar{e}^2)(1+U^2)^{1/2}[(1+U^2\bar{l}^2-1]\sin\bar{\Omega} - (\tfrac{2}{\bar{e}^2})(1+U^2)\cos\bar{\Omega}\cos\bar{i}$$

$$\text{W11} \equiv W_{11} = (2/\bar{e}^2)(1+U^2)\sin\bar{i}$$

EVALUATION OF EARTH ORBIT PROPERTIES AFTER ENCOUNTER

$$\text{PSI} \equiv \psi = \tan^{-1}[(V_{11}+1)/U_{11}]$$

$$\text{HE11} \equiv h_{e_{11}} = \bar{h} + V_{11} - 1/2$$

$$\text{QE11} \equiv q_{e_{11}} = -(V_{11}+1)V_{11} + 1$$

$$\text{PE11} \equiv p_{e_{11}} = (V_{11}+1)U_{11}$$

$$\text{XLE11} \equiv l_{e_{11}} = V_{11} + 1$$

CLOSEST APPROACH

$$\text{DBAR} \equiv \bar{d} = (1/2\bar{h})\left[(1 + 2\bar{h}\,\bar{l}^2)^{1/2} - 1\right]$$

COMPUTE PARAMETER CHECKS OF LAGERSTROM AND KEVORKIAN

$$\text{PARA1} = U(\gamma - \delta) = U(\text{CF11} - \text{CG11})$$

$$\text{PARA2} = \bar{l} + U T_1 - \lambda^2/2$$

EQUATIONS USED IN THE SUBROUTINES TO OBTAIN THE MOTION INTEGRALS

$$\text{TOOZ} \equiv t_{00}(z) = \left[\sin^{-1} 2\rho(z - \rho^2 z^2)^{1/2} - 2\rho(z - \rho^2 z^2)^{1/2}\right] \Big/ 2^{3/2}\rho^3$$

$$\text{TOOPZ} \equiv t_{00}'(z) = \left[z/2(1 - \rho^2 z)\right]^{1/2}$$

$$\text{ALPHAZ} \equiv \alpha(z) = -\left[z^{1/2}(1 - \rho^2 z)^{1/2}/2 + z^{1/2}/(1 - \rho^2 z)^{1/2} - \frac{3}{2\rho}\sin^{-1}\rho z^{1/2}\right] \Big/ 2^{1/2}\rho^4$$

$$\text{FOZ} \equiv f_0(z) = \frac{\cos[t_{00}(z) - T_0] - z}{\{1 + z^2 - 2z\cos[t_{00}(z) - T_0]\}^{3/2}} - \cos[t_{00}(z) - T_0]$$

$$\text{GOZ} \equiv g_0(z) = \frac{\sin[t_{00}(z) - T_0]}{\{1 + z^2 - 2z\cos[t_{00}(z) - T_0]\}^{3/2}} - \sin[t_{00}(z) - T_0]$$

$$\text{CF11} \equiv \gamma = \int_0^1 \left\{ f_0(z)[\alpha(1) - \alpha(z)] + \frac{1}{Q(1-z)} \right\} dz + \frac{1}{Q}$$

$$\text{CG11} \equiv \delta = \int_0^1 \left\{ z\, t_{00}'(z)\, g_0(z)[\beta(1) - \beta(z)] + \frac{1}{Q(1-z)} \right\} dz + \frac{1}{Q}$$

```
C
C
C
C
C
C
C THIS PROGRAM USES THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS TO SOLVE THE
C   SINGULAR PERTURBATION PROBLEM IN EARTH-MOON SPACE.  THE ANALYSIS IS BASED
C   ON WORK BY LAGERSTROM WHICH HAS APPEARED IN THE AIAA JOURNALS.
C
C
C
C          PROGRAM TO CALCULATE MOTIOM OF A BODY NEAR MOON IN THE           ORB00010
C          RESTRICTED 3-BODY PROBLEM                                        ORB00020
C                                                                           ORB00030
C                                                                           ORB00040
C          INITIAL DATA                                                     ORB00050
      IMPLICIT REAL*8(A-H,O-Z,$)                                            ORB00060
C      XM1 MASS PARENT,XM2 MASS TARGET                                      ORB00080
C      XMU RATIO OF MASSES                                                  ORB00090
C      PER- PERIOD OF SYSTEM                                                ORB00100
C      D - PLANET SEPERATION DISTANCE                                       ORB00110
C
      NAMELIST/DATA1/XM1,XM2,D                                              ORB00120
      NAMELIST/DATA3/RHO,XLAMB,XMU,T1
      NAMELIST/DATA2/XI,SOMEGA,J
      READ(5,DATA1)
      READ(5,DATA3)
    3 READ(5,DATA2)
      IF(J.NE.0)GO TO 999
C
C     COMPUTE PARAMETERS                                                    ORB00200
 5003 G=4.96720000E-10                                                      ORB00210
      IF(RHO.GT.1.D0)GO TO 990                                              ORB00220
       PER= 2.D0*3.1416D0*DSQRT((D**3)/(G*(XM1+XM2)))                       ORB00230
      COMMON/ADOLF/RHO,CAPTO,ALPHA1,Q,TOOP1,TOO1,Y1201,CAPT12,CAPT,U,N      ORB00240
      EXTERNAL SNT1                                                         ORB00260
      EXTERNAL SNT3                                                         ORB00290
C     COMPUTE ALL CONSTANTS TERMS FOR X=1   PART 1                          ORB00360
      TOOP1 = DSQRT(1.D0/(2.D0*(1.D0-(RHO**2))))                            ORB00370
      TOO1 = (DARSIN(2.D0*RHO*DSQRT(1.D0-(RHO**2)))-(2.D0*RHO*DSQRT         ORB00380
     A(1.D0-(RHO**2))))/((RHO**3)*(2.D0**(3.D0/2.D0)))                      ORB00390
      Y1201 = DSQRT(2.D0)*XLAMB*DSQRT(1.D0-(RHO**2))                        ORB00400
      CAPTO=TOO1                                                            ORB00410
      U = 1.D0/TOOP1                                                        ORB00440
       CAPT12=XLAMB*U*DCOS(XI)
        CAPT = CAPTO + DSQRT(XMU)*CAPT12 + XMU*T1                           ORB00430
      Q = (1.D0+U**2)**(3.D0/2.D0)                                          ORB00450
```

```
      ALPHA1 = -( (DSQRT(1.DO-(RHO**2))/2.DO)+ (1.DO/(DSQRT(1.DO-       ORB00460
     A(RHO**2)))) - (3.DO/(2.DO*RHO))*DARSIN(RHO))/((RHO**4)*           ORB00470
     B(DSQRT(2.DO)))                                                    ORB00480
      WRITE(6,99)                                                       ORB00490
 99   FORMAT('1','*****************************************************ORB00500
     A**********************'//)                                        ORB00510
      WRITE(6,100)                                                      ORB00520
      WRITE(6,2004)XM1,XM2,D,PER,G                                      ORB00530
 2004 FORMAT(1HO,T20,'MASS PARENT ='E12.6,'KGS',T50,'MASS TARGET='E12.6,ORB00540
     A'KGS',/,T20,'SEP DIS='E12.6,'KMS',T50,'PERIOD='F12.6,'DAYS'/      ORB00550
     B,T20,'GRAVITY CONSTANT='E12.6,'KMS**3/DAY**2*KG'/)                ORB00560
      WRITE(6,101)RHO,XMU,XLAMB,T1,XI,SOMEGA
      WRITE(6,103)TOOP1,TOO1,Y1201,CAPTO,CAPT12,CAPT,U,Q,ALPHA1         ORB00590
      WRITE(6,1000)                                                     ORB00600
      IF(RHO.GT.0.900D0) GO TO 4000                                     ORB00610
      CALL DQG16(0.DO,1.DO,SNT1,Y1)                                     ORB00620
      CF11 = Y1 + 1.DO/Q                                                ORB00640
      CALL DQG16(C.DO,1.DO,SNT3,Y3)                                     ORB00650
      CG11 = Y3 + 1.DO/Q                                                ORB00670
      GO TO 4001                                                        ORB00680
C
 4000 CALL DQG16(0.DO,1.DO,SNT5,Y5)                                     ORB00740
      GSS=(1.DO/RHO**4)*Y5                                              ORB00760
      CALL DQG16(0.DO,1.DO,SNT6,Y6)                                     ORB00770
      GS=Y6+(1.DO/((3.DO-2.DO*(RHO**2))**(3.DO/2.DO)))                  ORB00790
      CALL DQG16(C.DO,1.DO,SNT7,Y7)                                     ORB00800
      DS=Y7+(1.DO/((3.DO-2.DO*(RHO**2))**(3.DO/2.DO)))                  ORB00810
      CF11=(1.DO/U)*GSS+GS-(1.DO/((RHO**4)*(3.DO-2.DO*(RHO**2))         ORB00820
     A**(3.DO/2.DO)))*DLOG(U*(DSQRT((U**2)+2.DO)-U))                    ORB00830
      CG11=DS+(1.DO/((3.DO-2.DO*(RHO**2))**(3.DO/2.DO)))*DLOG(          ORB00840
     AU*(DSQRT((U**2)+2.DO)-U))                                         ORB00850
C
C
 4001 WRITE(6,104) CF11,CG11                                            ORB00860
C
C
C       COMPUTE FIRST ORDER MOTIOM ONLY                                 ORB00900
        HBAR=(1.DO+U**2)/2.DO                                           ORB00910
      ABAR=1.DO/(1.DO+U**2)                                             ORB00930
       SPSI10=0.DO
       ZETA10=XI*DSIN(SOMEGA)
      OMEGB=DATAN(-1.DO/U)
      XIBAR=DATAN((-ZETA10*DSQRT(1.DO+U**2))/(1.DO+(XLAMB**2)*
     1(1.DO+(U**2)*DSIN(XI)**2)/2.DO +U*(CF11-CG11)-U*(CAPTO+
     2T1)-SPSI10) )
      XLBAR=(1.DO+(XLAMB**2)*(1.DO+(U**2)*DSIN(XI)**2)/2.DO
     1+U*(CF11-CG11)-U*(CAPTO+T1)-SPSI10)/DCOS(XIBAR)
       PBAR=U*XLBAR+1.DO/DSQRT(2.DO*HBAR   )                            ORB00940
```

```
      QBAR=XLBAR-U/DSQRT(2.DO*HBAR)                                     ORBC0950
      EPSBAR=DSQRT(1.DO+2.DO*HBAR*(XLBAR**2))                           ORB00960
      THEBAR=DATAN(-XLBAR*DSQRT(1.DO+U**2))
      TAUBAR=(ABAR**(3.DO/2.DO))*DLOG(XMU*(ABAR**(3.DO/2.DO))
     1*EPSBAR*U/2.DO)+((1.DO-ABAR)/U)*XLBAR*DCOS(XIBAR)
     2+ABAR**(3.DO/2.DO)+T1+CG11
      THEBAR= THEBAR*(180.DO/3.1415927DO)                               ORB01010
      WRITE(6,850)                                                      ORB01020
      WRITE(6,851)HBAR,XLBAR,ABAR,PBAR,QBAR,EPSBAR,THEBAR,TAUBAR,OMEGB
     1,XIBAR
C     COMPUTR INCCMING VELOCITIES UI AND VI                             ORB01040
      UI=U
      VI=-1.DO
      WI=0.DO
C      COMPUTE OUTGOING COMPONENTS OF VELOCITY                          ORB01090
      UIIP=(1.DO/(EPSBAR**2))*DSQRT(1.DO+U**2)*
     1((1.DO+U**2)*(XLBAR**2)-1.DO)
      VIIP=(2.DO/(EPSBAR**2))*(1.DO+U**2)
      UII=UIIP*DCOS(OMEGB)-VIIP*DSIN(OMEGB)*DCOS(XIBAR)
      VII=UIIP*DSIN(OMEGB)+VIIP*DCOS(OMEGB)*DCOS(XIBAR)
      WII=VIIP*DSIN(XIBAR)
C     THE INITIAL DIRECTION AFTER MOON RELATIVE TO EARTH                ORB01160
      PSI=DATAN((VII+1.DO)/UII)                                         ORB01170
      PSID=PSI*180.DO/3.1415927DO                                       ORB01180
C      KEPLERIAN INTEGRALS FOR MOTION RELATIVE TO EARTH AFTER MOON PASSAORB01190
      HEII = HBAR+VII -0.5DO                                            ORB01200
       XLEII = VII + 1.DO                                               ORB01210
      PEII = UII*(VII+1.DO)                                             ORB01220
      QEII=-(VII+1.DO)*VII+1.DO                                         ORB01230
C     PRINT TARGET BODY DATA                                            ORB01240
      WRITE(6,3000)UI,VI,WI,UII,VII,WII
 3000 FORMAT(1H0,10X,'***** INCOMING AND OUTGOING VELOCITIES *****'/
     1,T5,'UI='E12.6,T23,'VI='E12.6,T40,'WI='E12.6,/T5,
     2'UII='E12.6,T23,'VII='E12.6,T40,'WII='E12.6/)
       WRITE(6,3001)PSID,HEII,XLEII,PEII,QEII                           ORBC1280
3001  FORMAT(1H0,T10,'PSIDEGREE='E12.6,T40,'HEII='E12.6,T60,'LEII='E12.6ORB01290
     A,/T10,'PEII='E12.6,T40,'QEII='E12.6,/)                            ORB01300
      DBAR=(1.DO/(2.DO*HBAR))*(DSQRT(1.DO+2.DO*HBAR*(XLBAR**2))-1.DO)   ORB01310
      DBAR=DBAR*XMU*D                                                   ORB01320
      DEII=(1.DO/(2.DO*HEII))*(1.DO-DSQRT(1.DO+2.DO*HEII*(XLEII**2)))   ORB01330
      DEIIKM=DEII*D                                                     ORB01340
      WRITE(6,3002)DBAR
      PARA1=U*(CF11-CG11)                                               ORB01360
      PARA2=XLBAR+U*T1-(XLAMB**2)/2.DO                                  ORB01370
      WRITE(6,3008)PARA1,PARA2                                          ORB01380
3008  FORMAT(1H0,'U(GAMMA-DELTA)='E15.8,5X,'LBAR+UT1-(LAMB**2)/2='E15.8)ORB01390
 3002 FORMAT(1H0,' CLOSEST APPROACH ='E15.8,/)
  850 FORMAT(1X,'                   ****TARGET ORBIT PARAMETERS****')   ORB01420
```

```
851    FORMAT(1HO,T10,' HBAR='E12.6,T30,' LBAR='E12.6,T50,' ABAR='E12.6/,ORB01430
      AT10,' PBAR='E12.6,T30,' QBAR='E12.6,T50,' EPSBAR='E12.6/,        ORB01440
      BT10,' THEBAR='E12.6,T30,' TAUBAR='E12.6,T50,'OMEGAB='E12.6,
      C/,T10,'IBAR='E12.6,/)
  104 FORMAT(1X,T10, ' GAM='E15.8,T50,' DEL='E15.8)                       ORB01460
1000   FORMAT(1X,T20,' ////////// THE INTEGRALS ARE //////////' /)        ORB01470
103    FORMAT(1HO,T10,'TOOP1='E12.6,T35,'TOO1='E12.6,T65,'Y1201='E12.6,/ ORBC1480
      AT10,'CAPTO='E12.6,T35,'CAPT12='E12.6,T65,'CAPT='E12.6,/          ORB01490
      BT10,'U='E12.6,T35,'Q='E12.6,T65,'ALPHA1='E12.6,/)                ORB01500
101    FORMAT(1HO,T20,'RHO='E12.6,T50,' MU='E12.6,/                       ORB01530
      AT20,' LAMBDA='E12.6,T50,' T1='E12.6,/,T20,' I='E12.6,T50,
      B' WSUB1='E12.6,/)
100    FORMAT(1HO,T30,' THE INITIAL PARAMETERS ARE ')                     ORB01550
       WRITE(6,995)                                                       ORB01560
995    FORMAT(1HO,'<<<<<<<<<<<<ALL FOR THIS CALCULATION >>>>>>>>>>>>>>') ORB01570
       GO TO 3
990    WRITE(6,991)                                                       ORB01590
991    FORMAT(1HO,' RHO GREATER THAN 1.DO STOPPING')                      ORB01610
  999 STOP                                                                ORB01620
       END                                                                ORB01630
C                                                                         ORB01640
C      ...........................................................................ORB01650
C                                                                         ORB01660
C         SUBROUTINE DQG16                                                ORB01670
C                                                                         ORB01680
C         PURPOSE                                                         ORB01690
C            TO COMPUTE INTEGRAL(FCT(X), SUMMED OVER X FROM XL TO XU)     ORB01700
C                                                                         ORB01710
C         USAGE                                                           ORB01720
C            CALL DQG16 (XL,XU,FCT,Y)                                     ORB01730
C            PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT                 ORB01740
C                                                                         ORB01750
C         DESCRIPTION OF PARAMETERS                                       ORB01760
C            XL     - DOUBLE PRECISION LOWER BOUND OF THE INTERVAL.       ORB01770
C            XU     - DOUBLE PRECISION UPPER BOUND OF THE INTERVAL.       ORB01780
C            FCT    - THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION   ORB01790
C                     SUBPROGRAM USED.                                    ORB01800
C            Y      - THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.      ORB01810
C                                                                         ORB01820
C         REMARKS                                                         ORB01830
C            NONE                                                         ORB01840
C                                                                         ORB01850
C         SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                   ORB01860
C            THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)     ORB01870
C            MUST BE FURNISHED BY THE USER.                               ORB01880
C                                                                         ORB01890
C         METHOD                                                          ORB01900
C            EVALUATION IS DONE BY MEANS OF 16-POINT GAUSS QUADRATURE     ORB01910
```

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C          FORMULA, WHICH INTEGRATES POLYNOMIALS UP TO DEGREE 31          ORB01920
C          EXACTLY. FOR REFERENCE, SEE                                    ORB01930
C          V.I.KRYLOV, APPROXIMATE CALCULATION OF INTEGRALS,              ORB01940
C          MACMILLAN, NEW YORK/LONDON, 1962, PP.100-111 AND 337-340.      ORB01950
C                                                                         ORB01960
C     ....................................................................ORB01970
C                                                                         ORB01980

      SUBROUTINE DQG16(XL,XU,FCT,Y)                                       ORB01990
C                                                                         ORB02000
C                                                                         ORB02010
      DOUBLE PRECISION XL,XU,Y,A,B,C,FCT                                  ORB02020
C                                                                         ORB02030
      A=.5D0*(XU+XL)                                                      ORB02040
      B=XU-XL                                                             ORB02050
      C=.4947004674958249 7D0*B                                           ORB02060
      Y=.1357622970587704 7D-1*(FCT(A+C)+FCT(A-C))                        ORB02070
      C=.4722875115366162 9D0*B                                           ORB02080
      Y=Y+.3112676196932394 6D-1*(FCT(A+C)+FCT(A-C))                      ORB02090
      C=.4328156011939158 7D0*B                                           ORB02100
      Y=Y+.4757925584124639 2D-1*(FCT(A+C)+FCT(A-C))                      ORB02110
      C=.3777022041775015 2D0*B                                           ORB02120
      Y=Y+.6231448562776693 6D-1*(FCT(A+C)+FCT(A-C))                      ORB02130
      C=.3089381222013218 7D0*B                                           ORB02140
      Y=Y+.7479799440828837D-1*(FCT(A+C)+FCT(A-C))                        ORB02150
      C=.2290083888286136 9D0*B                                           ORB02160
      Y=Y+.8457825969750127D-1*(FCT(A+C)+FCT(A-C))                        ORB02170
      C=.1408017753896294 6D0*B                                           ORB02180
      Y=Y+.9130170752246179D-1*(FCT(A+C)+FCT(A-C))                        ORB02190
      C=.4750625491881872D-1*B                                            ORB02200
      Y=B*(Y+.9472530522753425D-1*(FCT(A+C)+FCT(A-C)))                    ORB02210
      RETURN                                                              ORB02220
      END                                                                 ORB02230

      FUNCTION SNT1(Z)                                                    ORB02240
      IMPLICIT REAL*8(A-H,O-Z,$)                                          ORB02250
      COMMON/ADOLF/RHO,CAPTO,ALPHA1,Q,TOOP1,TOO1,Y1201,CAPT12,CAPT,U,N    ORB02260
      N=N+1                                                               ORB02270
      TOOZ =    (DARSIN(2.D0*RHO*DSQRT(Z-(RHO**2)*(Z**2))) - 2.D0 * RHO*  ORB02280
     ADSQRT(Z-(RHO**2)*(Z**2)))/((RHO**3)*(2.D0**(3.D0/2.D0)))            ORB02290
      ALPHAZ=-((((DSQRT(Z))*DSQRT(1.D0-(RHO**2)*Z))/2.D0) +(DSQRT(Z)/     ORB02300
     A(DSQRT(1.D0-(RHO**2)*Z))) -((3.D0/(2.D0*RHO))*DARSIN(RHO*DSQRT      ORB02310
     B(Z))))/((RHO**4)*DSQRT(2.D0))                                       ORB02320
      FOZ=((DCOS(TOOZ-CAPTO)-Z)/((1.D0+(Z**2)-2.D0*Z*DCOS(TOOZ-           ORB02330
     ACAPTO))**(3.D0/2.D0)))-DCOS(TOOZ-CAPTO)                             ORB02340
      SNT1=FOZ*(ALPHA1-ALPHAZ)+(1.D0/(Q*(1.D0-Z)))                        ORB02350
```

```
      RETURN                                                            ORB02360
      END                                                               ORB02370

      FUNCTION SNT3(Z)                                                  ORB02480
      IMPLICIT REAL*8(A-H,O-Z,$)                                        ORB02490
      COMMON/ADOLF/RHO,CAPTO,ALPHA1,Q,TOOP1,TOO1,Y1201, CAPT12,CAPT,U,N ORB02500
      N=N+1                                                             ORB02510
      BETA1 = -(1.DO/TOOP1)                                             ORB02520
      TOOPZ = DSQRT(Z/(2.DO*(1.DO-(RHO**2)*Z)))                         ORB02530
      BETAZ = -(1.DO/TOOPZ)                                             ORB02540
      TOOZ =    (DARSIN(2.DO*RHO*DSQRT(Z-(RHO**2)*(Z**2))) - 2.DO * RHO* ORB02550
     ADSQRT(Z-(RHO**2)*(Z**2)))/((RHO**3)*(2.DO**(3.DO/2.DO)))         ORB02560
      GOZ = (DSIN(TOOZ-CAPTO)/((1.DO+(Z**2)-2.DO*Z*DCOS(TOOZ-CAPTO))**  ORB02570
     A(3.DO/2.DO)))- DSIN(TOOZ-CAPTO)                                   ORB02580
      SNT3 = Z*TOOPZ*GOZ*(BETA1-BETAZ) + (1.DO/(Q*(1.DO-Z)))            ORB02590
      RETURN                                                            ORB02600
      END                                                               ORB02610
```

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13. ABSTRACT

A program was written for the IBM 360 system of the Naval Postgraduate School Computer Facility to apply the methods of singular perturbation theory to earth-moon trajectories.

Verification of results against previous work was carried out and it was found that agreement could be attained only for energy parameter values of 0.707 or less. No solution for higher values could be found.

The analysis of three dimensional orbits was then conducted within this restricted range to show the merit of singular perturbation theory as an initial design tool.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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